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Object-Oriented Modelling and Simulation of Flexible Multibody Thin Beams in Modelica with the Finite Element Method

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Abstract

In this paper the development, simulation and validation of *Modelica* models for flexible thin beams is presented.
The models are based on the application of the finite element method. Exploiting the object-oriented features of the language, mixed-mode models (finite element-finite volume) are developed as well.
All the models use the standard connectors defined within the *Modelica* multibody library, guaranteeing thus full compatibility with the library components.
The details of the mathematical modelling are fully analyzed, showing the development of the equations of motion.
The models feature also a graphical interface, with visualization of the simulation outcomes within the same 3D environment used in the multibody library, allowing the user to have an immediate visual feedback.
Finally, the models are analyzed and validated by mean of selected simulation experiments, with reference both to theoretical predictions and to results commonly accepted within the scientific literature.

1 Introduction

Many engineering applications require the development of simulation models for flexible multibody systems (e.g., robot manipulators, helicopter rotors, aircraft wings, space structures, machining tools, car suspensions, etc.) both dynamically accurate and computationally affordable.
The task of developing models for generic-shaped, fully deformable bodies is usually demanded to specialized simulation codes and tools, due to the complexity of the task. Such models are usually adequate for structural analysis and design tasks, while being far too complex for affordable dynamics simulation and analysis.
On the other hand, particular classes of deformable bodies, such as flexible beams, can be represented with less complex models which are still able to represent all the dynamically relevant deformation effects.
Flexible beams are continuous non linear dynamical systems characterized by an infinite number of degrees of freedom. Obviously, dealing directly with infinite dimensional models is impractical both for dynamic analysis and simulation purposes. Hence it is necessary to introduce methods to describe flexibility with a discrete number of parameters.
Three different approaches have been traditionally used to derive approximated finite dimensional models: lumped parameters, assumed modes and finite element method [3],[5].
The lumped parameter approach is the simplest one. In this method each flexible beam is divided into a finite number of rigid beams, introducing pseudojoints, and the flexibility is represented by springs that restrict the motion of each pseudojoint. This method is however rarely used because of the difficulty in determining the spring constants of the pseudojoints and then of achieving a suitable accuracy up to the desired approximation frequency.
The assumed modes model formulation has been widely used in the literature [6]. It describes beam flexibility using truncated modal series, based on spatial mode eigenfunctions and time varying vibrational modes. One of the best features offered by such a method is the fine control on the accuracy up to the desired approximation frequency. Although conceptually simple, this description requires to find out the best selection for spatial modal shapes and the boundary conditions, which is not at all a trivial task. In addition to that, the selection of the appropriate eigenfunc-

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tions and the resulting vibrational modes could depend on the boundary conditions for the specific case at hand, ruling thus out the possibility of a modular approach for the model development.
In the finite element method approach [9], the flexible beam is divided into several elements, with a local description of the deformation field by the use of element-wise basis functions. Although such approach could be computationally more demanding than the modal one (it is usually necessary to use a larger number of elements than of modal eigenfunctions to obtain the same approximation), it allows a formulation which is independent of the actual boundary condition [7]. The finite element method is then a viable choice for the representation of flexible beams within a modular environment.
As far as the theory of elasticity to be used is concerned, it must be pointed out that beam deflection, with respect to the rigid configuration, is generally assumed to be small, which allows to adopt linear theory. In this case the Euler-Bernoulli theory [8] can be used to describe beam flexibility, neglecting the effects of shear deformation and assuming uniform cross-sectional properties along the beam. In this paper, we consider linear elasticity theory for the modelling of flexible *thin* beams. On the other side, Timoshenko theory [8] should be used for models where such effects need to be taken into account (e.g., for short beams).
The paper is organized as follows: in Section 2 the problem of the representation of a generic deformable body in a multibody system is introduced; in Section 3 the development of the equations of motions is shown, with reference both to the finite element method case and to the mixed-method one; in Section 4 the *Modelica* implementation is analyzed; Section 5 contains selected simulation results; finally, in Section 6 the main results are summarized and future developments are introduced.

2 Deformable Body Degrees of Freedom

Consider a generic multibody system (Fig. 1). The position, in body coordinates, of a point on a specific deformable body has the following expression:

$$\overline{u} = \overline{u}_0 + \overline{u}_f\,, \tag{1}$$

where $\overline{u}_0$ is the “undeformed” (i.e., rigid) position vector and $\overline{u}_f$ is the deformation contribution to position (i.e., the deformation field).

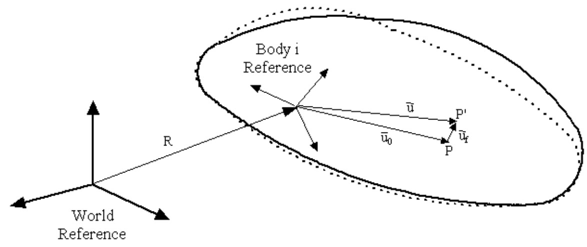

Figure 1: Flexible body reference systems

The formal and mathematically sound description of the generic deformation of a body requires the deformation field to belong to an infinite dimensional functional space, requiring, in turn, an infinite number of deformation degrees of freedom.
In this paper, the deformation field is described by an approximation of the functional basis space it belongs to, supposing such space has a finite dimension, say M, so that the vector u_f can be expressed by the following finite dimensional product:

$$\overline{u}_f = S q_f\,, \tag{2}$$

where S is the $[3 \times M]$ shape functions matrix (i.e., a matrix of functions defined over the body domain and used as a basis to describe the deformation field of the body itself) and q_f is the M-dimensional vector of deformation degrees of freedom.
The position of a point on a deformable body can then be expressed in world reference as follows:

$$r = R + A\overline{u} = R + A(\overline{u}_0 + Sq_f) = R + A\overline{u}_0 + ASq_f\,, \tag{3}$$

where R is the vector identifying the origin of the body local reference system and A is the rotation matrix for the body reference system.
The representation of a generic deformable body in world reference requires then $6+M$ d.o.f. (i.e., 6 corresponding to rigid displacements and rotations and M to deformation fields):

$$q = [q_r\ \ q_f]^T = [R\ \ \theta\ \ q_f]^T\,, \tag{4}$$

where θ represents the undeformed body orientation angles and q_r is a vector containing the 6 rigid degrees of freedom.

3 Motion Equations

The equations of motion for a generic flexible body in a multibody system can be developed applying the principle of virtual work [3]. It should be pointed out that the same results could be obtained using the classical Lagrangian approach (as in, e.g., [5]), though such approach is quite knotty and difficult to use in practice, due to the complexity of the required analytical differentiation of the kinetic energy expression.

The principle of virtual work states that the virtual work of the inertial forces δW_i must counterbalance the sum of the virtual work of the *continuum* elastic forces δW_s and of the external ones δW_e:

$$\delta W_i = \delta W_s + \delta W_e \; . \tag{5}$$

Note that, in case $\delta W_i = 0$, the problem reduces to the well-known problem of structural statics [9].

The terms of equation (5) are defined as follows:

$$\begin{aligned} \delta W_i &= \int_V \rho \delta r^T \ddot{r} dV \; , & (6)\\ \delta W_s &= -\int_V \delta \varepsilon^T \sigma dV \; , & (7)\\ \delta W_e &= \int_V \delta r^T F_e \, dV + \int_\Omega \delta r^T f_e \, d\Omega \; , & (8) \end{aligned}$$

where V is the body volume, ρ is the body density, δr is an infinitesimal virtual displacement, $\ddot{r}$ is the body acceleration (in world reference), $\delta\varepsilon$ is a vector of virtual infinitesimal internal strains, σ is the internal stresses vector, F_e is the vector of external volume forces, Ω is the body surface and f_e is the vector of external surface forces.

The quantities δr and $\ddot{r}$ can be computed using equation (3):

$$\begin{aligned} \delta r &= \delta R + \delta A\bar{u} + A\delta\bar{u} = \delta R + \theta_d \times A\bar{u} + AS\delta q_f \; , \\ \ddot{r} &= \ddot{R} + \omega \times \omega \times u + \alpha \times u + 2\omega \times A\dot{\bar{u}} + A\ddot{\bar{u}} \; , \end{aligned} \tag{9}$$

where α and ω are the body angular acceleration and velocity (in world reference),respectively, and $\theta_d = \omega dt$ represents a virtual-infinitesimal rotation.

The expressions in (9) can be substituted in 5, leading to

$$\delta W_i = \int_V \rho \delta r^T \ddot{r} dV = \delta R^T Q_i^R + \theta_d^T Q_i^\theta + \delta q_f^T Q_i^f \; . \tag{10}$$

The terms Q_i^R,Q_i^θ and Q_i^f can be calculated using the following definitions:

$$\begin{aligned} m_{RR} &= \int_V \rho \, dV \; , & (11)\\ m_{R\theta} &= \int_V \rho A \left(\bar{u}\times\right)^T A^T dV \; , & (12)\\ m_{Rf} &= \int_V \rho AS \, dV \; , & (13)\\ m_{\theta\theta} &= -\int_V \rho A\bar{u} \times \bar{u} \times A^T \, dV \; , & (14)\\ m_{\theta f} &= \int_V \rho A\bar{u} \times S \, dV \; , & (15)\\ m_{ff} &= \int_V \rho S^T S \, dV \; , & (16)\\ \bar{S} &= \int_V \rho S dV = A^T m_{Rf} \, , & (17)\\ \bar{S}_t &= \int_V \rho \bar{u} dV \, , & (18)\\ \tilde{\bar{S}}_t &= \int_V \rho \left(\bar{u}\times\right) dV = A m_{R\theta} A^T \; , & (19)\\ \bar{I}_{\theta\theta} &= \int_V \rho \left(\bar{u}\times\right)^T \left(\bar{u}\times\right) dV = A^T m_{\theta\theta} A \, , & (20)\\ \bar{I}_{\theta f} &= \int_V \rho \left(\bar{u}\times\right) S dV = A^T m_{\theta f} \, . & (21) \end{aligned}$$

The vector Q_i^R can then be obtained as follows:

$$\begin{aligned} Q_i^R = & \int_V \rho \ddot{R} dV + \int_V \rho\omega \times (\omega \times u) \, dV + \int_V \rho \left(\alpha \times u\right) dV \\ & + \int_V \rho 2\omega \times \left(A\dot{\bar{u}}\right) dV + \int_V \rho A\ddot{\bar{u}} dV = \\ & = m_{RR}\ddot{R} + A\bar{\omega} \times \bar{\omega} \times A^T \int_V \rho A\bar{u} dV + \\ & + A\bar{\alpha} \times A^T \int_V \rho A\bar{u} dV + 2A\bar{\omega} \times A^T \int_V \rho AS dV \dot{q}_f \\ & + \int_V \rho AS dV \ddot{q}_f = \\ & = m_{RR}\ddot{R} + A\tilde{\bar{S}}_t^T \bar{\alpha} + A\bar{S}\ddot{q}_f + A\left(\bar{\omega} \times \bar{\omega} \times \bar{S}_t + 2\bar{\omega} \times \bar{S}\dot{q}_f\right) = \\ & = m_{RR}\ddot{R} + m_{R\theta}\alpha + m_{Rf}\ddot{q}_f - AQ_v^R \, , \end{aligned} \tag{22}$$

being $Q_v^R = -\bar{\omega} \times \bar{\omega} \times \bar{S}_t - 2\bar{\omega} \times \bar{S}\dot{q}_f$ the quadratic velocity vector (due to Coriolis and centrifugal forces) associated to translational degrees of freedom.

The second term of the generalized inertial forces can be expressed as

$$
\begin{aligned}
Q_i^\theta =& A\int_V \rho\,(\bar{u}\times)\,dV\,A^T\ddot{R} - \omega\times\int_V \rho u\times u\times dV\,\omega \\
&-\int_V \rho u\times u\times dV\,\alpha + 2\int_V \rho u\times\omega\times(AS\dot{q}_f)\,dV \\
&+\int_V \rho u\times(AS\ddot{q}_f)\,dV = \\
&= A\tilde{\tilde{S}}_t A^T\ddot{R} + A\bar{\omega}\times\int_V -\rho\bar{u}\times\bar{u}\times dV\bar{\omega} \\
&- A\int_V \rho\bar{u}\times\bar{u}\times dV\,\bar{\alpha} - 2A\int_V \rho\bar{u}\times(S\dot{q}_f)\times\bar{\omega}dV \\
&+ A\int_V \rho\bar{u}\times S dV\,\ddot{q}_f = \\
&= A\left(\tilde{\bar{S}}_t A^T\ddot{R} + \bar{\omega}\times\bar{I}_{\theta\theta}\bar{\omega} + \bar{I}_{\theta\theta}\bar{\alpha} + \dot{\bar{I}}_{\theta\theta}\bar{\omega} + \right. \\
&\left. + \bar{\omega}\times\bar{I}_{\theta f}\dot{q}_f + \bar{I}_{\theta f}\ddot{q}_f\right) = \\
&= m_{R\theta}^T\ddot{R} + m_{\theta\theta}\alpha + m_{\theta f}\ddot{q}_f - AQ_v^\theta,
\end{aligned}
\tag{23}
$$

where the quadratic velocity vector associated to the rotational degrees of freedom is $Q_v^\theta = -\bar{\omega}\times\bar{I}_{\theta\theta}\bar{\omega} - \dot{\bar{I}}_{\theta\theta}\bar{\omega} - \bar{\omega}\times\bar{I}_{\theta f}\dot{q}_f$.
The Q_i^f term, which is related to the deformation d.o.f. q_f, can be expanded as follows:

$$
\begin{aligned}
Q_i^f =& \int_V \rho S^T A^T\ddot{R}dV + \int_V \rho S^T A^T\omega\times(\omega\times u)\,dV \\
&+\int_V \rho S^T A^T(\alpha\times u)\,dV + \int_V \rho S^T A^T 2\omega\times\left(A\dot{\bar{u}}\right)dV \\
&+\int_V \rho S^T\ddot{u}dV = \bar{S}^T A^T\ddot{R} + \int_V \rho S^T\bar{\alpha}\times\bar{u}dV + \\
&+\int_V \rho S^T(\bar{\omega}\times\bar{\omega}\times\bar{u} + 2\bar{\omega}\times S\dot{q}_f)\,dV \\
&+\int_V \rho S^T S dV\,\ddot{q}_f = \\
&= \bar{S}^T A^T\ddot{R} + \bar{I}_{\theta f}^T\bar{\alpha} + m_{ff}\ddot{q}_f + \\
&+\int_V \rho S^T\left(\tilde{\bar{\omega}}^2\bar{u} + 2\tilde{\bar{\omega}}S\dot{q}_f\right)dV = \\
&= m_{Rf}^T\ddot{R} + m_{\theta f}^T\alpha + m_{ff}\ddot{q}_f - Q_v^f,
\end{aligned}
\tag{24}
$$

being $Q_v^f = -\int_V \rho S^T\left(\tilde{\bar{\omega}}^2\bar{u} + 2\tilde{\bar{\omega}}S\dot{q}_f\right)dV$.
The virtual work of the internal elastic forces, under the hypothesis of elastic constitutive law for the material, can be expressed as:

$$
\delta W_s = -\int_V \delta\varepsilon^T\sigma dV = -\delta q_f^T K_{ff} q_f\,, \tag{25}
$$

where K_{ff} represents the structural stiffness matrix. The form of such matrix depends on the specific material constitutive law and on the body shape.
The virtual work of external forces reads as follows:

$$
\delta W_e = \delta R^T Q_e^R + \theta_d^T Q_e^\theta + \delta q_f^T Q_e^f\,, \tag{26}
$$

where Q_e^R, Q_e^θ and Q_e^f represent, respectively, the generalized components of the active forces associated to translational, rotational and deformation coordinates.

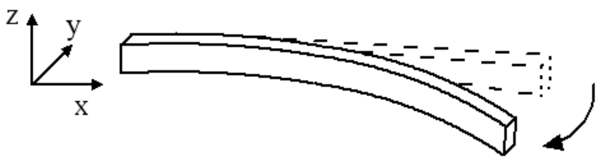

Figure 2: Planar beam deformation

Equation (5) must be satisfied for every virtual displacement so that the following identities must hold:

$$
\begin{aligned}
Q_i^R &= Q_e^R, & (27)\\
Q_i^\theta &= Q_e^\theta, & (28)\\
Q_i^f &= -K_{ff}q_f + Q_e^f. & (29)
\end{aligned}
$$

Equations (27), (28) and (29) are the equations for 3D motion of a generic flexible body characterized by an elastic constitutive law for its material. In the scientific literature, such expressions are generally referred to as the *generalized Newton-Euler* equations (see e.g., [5]). The equations of motion can be easily expressed in body axes, resulting in:

$$
\begin{aligned}
&\begin{bmatrix} m_{RR} & \tilde{\bar{S}}_t^T & \bar{S} \\ & \bar{I}_{\theta\theta} & \bar{I}_{\theta f} \\ & & m_{ff} \end{bmatrix}
\begin{bmatrix} \ddot{\bar{R}} \\ \bar{\alpha} \\ \ddot{q}_f \end{bmatrix} = \\
&= \begin{bmatrix} O_3 \\ O_3 \\ -K_{ff}q_f \end{bmatrix} + \begin{bmatrix} Q_v^R \\ Q_v^\theta \\ Q_v^f \end{bmatrix} + \begin{bmatrix} \bar{Q}_e^R \\ \bar{Q}_e^\theta \\ Q_e^f \end{bmatrix}.
\end{aligned}
\tag{30}
$$

Equations (30) are valid for a general deformable body, though many of the quantities involved (e.g., the matrix K_{ff}) depend on specific body characteristics such as the shape or the material properties.

From now on, the case of a *thin beam* will be considered. In detail, it will be assumed that the body is a 1D elastic *continuum* with constant cross-sectional properties. Furthermore, it will be assumed that the beam constitutive material is homogeneous, isotropic and perfectly elastic (i.e., the elastic internal forces are conservative). Finally, it will be assumed that the deformation field is restricted to lie within the *xy* plane of the beam local reference system (Fig. 2).

It should be pointed out that such assumptions do not restrict the model validity or generality, since the model remains still representative for a large number of dynamic simulation applications (e.g., almost all the flexible robots commonly studied have flexible links which can be represented by such model [7]).

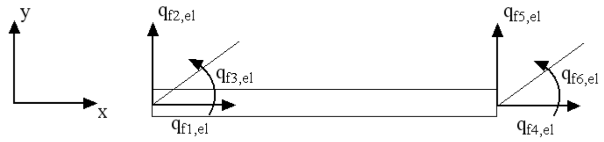

Figure 3: Element coordinate systems

3.1 The element point of view

The finite element method is based upon a discretization of the beam into N elements. A single element can itself be viewed as a thin beam characterized by a planar deformation field. It is then possible to define the local dimensionless *abscissa* as $\xi = x/\ell$, where x is the longitudinal local coordinate and ℓ is the element length.

In [9] it is shown that the partial differential equations associated with the deformation problem at hand, under the hypothesis of elastic constitutive law for the material, require, for a consistent finite element formulation, the use of linear and Hermite cubic polynomials for the approximation of the axial and transversal deformation field, respectively. Thus, for a single element, the generic equations of motion (30) can be expanded as follows:

$$\begin{aligned}
&\overline{u}_{f,el} = \begin{bmatrix} \overline{u}_{f1,el} \\ \overline{u}_{f2,el} \\ \overline{u}_{f3,el} \end{bmatrix} = S_{el}\, q_{f,el}\,, \\
&S_{el} = \begin{bmatrix} 1-\xi & 0 & 0 \\ 0 & 1-3\xi^2+2\xi^3 & \ell(\xi-2\xi^2+\xi^3) & \cdots \\ 0 & 0 & 0 \end{bmatrix} \\
&\begin{bmatrix} \cdots & \xi & 0 & 0 \\ \cdots & 0 & 3\xi^2-2\xi^3 & \ell(\xi^3-\xi^2) \\ & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} S_{el1} \\ S_{el2} \\ S_{el3} \end{bmatrix}, \\
&q_{f,el} = \begin{bmatrix} q_{f1,el} & q_{f2,el} & q_{f3,el} & q_{f4,el} & q_{f5,el} & q_{f6,el} \end{bmatrix}^T ,
\end{aligned} \tag{31}$$

where the subscript $_{el}$ is used to refer the quantities to a single element.

Fig. 3 depicts the element coordinate systems associated with the deformation degrees of freedom: $q_{f1,el}$ and $q_{f4,el}$ are associated with axial compression, $q_{f2,el}$ and $q_{f5,el}$ with transversal displacement and $q_{f3,el}$ and $q_{f6,el}$ with beam extremities rotation.

Since the third row of the shape matrix $S_{el}$is composed only by zeros, it could be noted that, despite the fact that the motion equations have been developed for a general 3D case, the deformation field is assumed to lie within the local xy plane.

The planar deformation hypothesis and the assumption of a homogeneous, isotropic and elastic material for the beam, allow to exploit the Euler-Bernoulli theory and to calculate the elastic potential energy U_{el}, neglecting the contribution of shear stresses and considering only the work of the resulting axial force N_{el} and bending moment M_{el}, as follows [9]:

$$\begin{aligned}
U_{el} = &\frac{1}{2}\int_\ell \left(\frac{N_{el}\,N'_{el}}{EA} + \frac{M_{el}\,M'_{el}}{EJ} \right) dx = \\
= &\frac{1}{2}\int_\ell \left(EJ\overline{u}''^2_{f2,el} + EA\overline{u}'^2_{f1,el} \right) dx = \frac{1}{2} q^T_{f,el} K_{ff,el} q_{f,el}\,,
\end{aligned} \tag{32}$$

where E is the material Young's modulus, A is the (constant) cross-sectional area and J is the (constant) cross-sectional second moment of area. The analytical expression for the case at hand for the matrix $K_{ff,el}$, usually known as the structural stiffness matrix, is reported in appendix A.

3.2 Finite Element Method Equations Assembly

The equations of motion for the entire beam can be obtained by assembling the equations of motion for beam elements as the one defined in the previous subsection. The body reference system will be the local reference system located at the root of the first element, so that the rigid degrees of freedom, common to all the elements, will be referred to such coordinate system.

Let then m and L be the mass and length of the entire beam, and N the number of elements to be used, so that $\ell = L/N$. Indicating with $\widehat{\overrightarrow{X}}$ the reference system unit vector along the beam axis, the expression of the generic position $\overline{u}_j$ of a point of element j can be expressed as:

$$\overline{u}_j = \overline{u}_{0j} + S_{el}B_j q_f = [\xi_j\ell + (j-1)\ell]\,\widehat{\overrightarrow{X}} + S_{el}B_j q_f\,, \tag{33}$$

where $\overline{u}_{0j}$ is the position of the root of the j^{th} element, S_{el} is the shape functions matrix defined by (31), B_j is the so-called *connectivity matrix* and q_f is a vector containing the deformation degrees of freedom for the whole beam.

The matrices B_j have the following form:

$$B_j = \left[\ O_{6,3(j-1)}\ \middle|\ I_6\ \middle|\ O_{6,3(N-j)}\ \right], \forall j = 1,\cdots,N\,. \tag{34}$$

The connectivity matrices are used to relate the vector q_f, which contains the deformation degrees of freedom for the whole beam, to the corresponding j^{th} element, according to the expression:

$$q_{f,el_j} = B_j q_f\,. \tag{35}$$

The dynamics of the complete flexible beam can then be described by equation (30), using the following expressions:

$$\begin{aligned}
\bar{S} &= \sum_{j=1}^{N} \frac{m}{L}\int_{V_j} S_{el} B_j dV_j \,,\\
\bar{S}_t &= \sum_{j=1}^{N} \frac{m}{L}\int_{V_j} \bar{u}_j dV_j \,,\\
\bar{I}_{\theta\theta} &= \sum_{j=1}^{N} \frac{m}{L}\int_{V_j} \begin{pmatrix} \bar{u}^2_{2f_j} & -\bar{u}_{2f_j}\bar{u}_{1_j} & 0 \\ & \bar{u}^2_1 & 0 \\ & & \bar{u}^2_{1_j}+\bar{u}^2_{2f_j} \end{pmatrix} dV_j \,,\\
\bar{I}_{\theta f} &= \sum_{j=1}^{N} \frac{m}{L}\int_{V_j} \begin{pmatrix} O_{(3N,1)} \\ O_{(3N,1)} \\ \bar{u}_{1_j} S_{el2} - \bar{u}_{2_j} S_{el2} \end{pmatrix} dV_j \,,\\
m_{ff} &= \sum_{j=1}^{N} \frac{m}{L} B_j^T \left(\int_{V_j} S_{el}^T S_{el} dV_j\right) B_j \,,\\
K_{ff} &= \sum_{j=1}^{N} B_j^T K_{ff,el} B_j \,,\\
Q_v^f &= -\sum_{j=1}^{N} \frac{m}{L}\int_{V_j} \left[B_j^T S_{el}^T \left(\tilde{\bar{\omega}}^2 \bar{u}_j + 2\tilde{\bar{\omega}} S_{el} B_j \dot{q}_f\right)\right] dV_j \,.
\end{aligned} \tag{36}$$

The computation of the above terms can be easily carried out by observing that the integral of a generic quantity $\mathcal{F}$, varying along the beam, onto the volume of a single element can be computed as follows:

$$\int_{V_j} \rho \mathcal{F} dV_j = \frac{m}{L}\int_0^1 \ell \mathcal{F}(\xi) d\xi = \frac{m}{N}\int_0^1 \mathcal{F}(\xi) d\xi \,. \tag{37}$$

3.3 Boundary Conditions

The equations of motion for the whole beam must be completed by enforcing suitable boundary conditions for the finite element approximation of the deformation partial differential equations. That means assuming prescribed values for some of the deformation displacements, rotations and velocities (linear or angular) at the body boundaries which are, for the case at hand, the beam root and tip.

The most commonly used boundary conditions for flexible beams are of two kinds, commonly referred to as *clamped-free* and *simply-supported* conditions. In both cases six conditions are given (as it is required from the underlying partial differential equations): the *clamped-free* ones enforce null deformation at the beam root (i.e., q_{f1}, q_{f2}, q_{f3}, $\dot{q}_{f1}$, $\dot{q}_{f2}$, $\dot{q}_{f3}$ equal to zero for the first element), while the *simply-supported* ones enforce null axial and transversal displacement at the beam root (i.e., q_{f1}, q_{f2}, $\dot{q}_{f1}$, $\dot{q}_{f2}$ equal to zero for the first element) and transversal displacement at the beam tip (i.e., q_{f5} and $\dot{q}_{f5}$ equal to zero for the last element).

The choice of which of the two set of conditions has to be used largely depends on the problem at hand.

It should be pointed out that the boundary conditions names are just conventional and are not referred to the objects the beam is connected or linked to (e.g., joints or other bodies), so that enforcing such boundary condition does not limitate in any way the generality and modularity of the model developed so far.

The enforcement of the boundary conditions is traditionally obtained by introducing suitable matrices in equations (30) [6, 9]. On the other hand, it can be observed that such conditions can be enforced by suitable modifications of the connectivity matrices B_1 and B_N, by zeroing some entries. For example, for the *clamped-free* conditions, B_N remains unvaried and B_1 becomes

$$B_1 = \left[\begin{array}{cc|c} 0_3 & 0_3 & \\ 0_3 & I_3 & O_{6,3(N-1)} \end{array}\right] . \tag{38}$$

3.4 Extended Formulation of the Equation of Motion

In the finite element formulation for the equation of motion for a flexible beam, the reference directions of the internal actions are the same for all the elements. Such representation is acceptable as long as the deformation field is small compared to the beam length, as it is the case, for example, when studying the dynamics of vibrations in machining tools.

On the other hand, when large deformations are involved, the internal actions reference directions should change according to the deformation field. That means that it is necessary to define a local reference system for each element (Fig. 4). This corresponds to the application of the finite volume method to assemble the equations of motion solved over each element (i.e., over each volume). This representation is valid also for large beam deformation, as long as the deformation field is small compared to the volumes length.

Furthermore, it is possible to assemble the equation of motion for a mixed (finite element-finite volume) formulation by dividing every volume into several elements.

It is not necessary to go into the detailed calculations for the finite volume or the mixed formulation since, as it will be shown in section 4, the equations of motion for such extensions can be automatically calculated with the aid of symbolic manipulation algorithms applied to the finite element formulation.

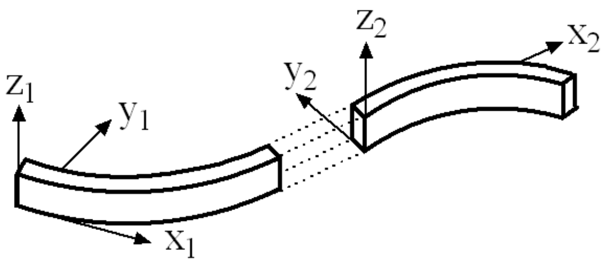

Figure 4: Volume coordinate systems

4 Modelica Implementation

The finite element formulation for the model has been implemented using the *Modelica* language, creating thus a new component, called *FlexBeamFEM* (Fig. 5). The component interfaces are two standard mechanical flanges from the new *MultiBody* library [4]. The connectors choice makes the component fully compatible with the library, so that it is possible to connect directly the flexible beam component with the predefined models such as mechanical constraints (revolute joints, prismatic joints, etc.), parts (3D rigid bodies) and forces elements (springs, dampers, forces, torques).

Figure 5: Component icon

In detail, the flexible beam component uses two mechanical flanges as physical representation of the two ends of the beam while the motion is ruled by equations (30), with addition of a damping term $(-D_{ff}\dot{q}_f)$ for the structural dynamics part. The damping term is added to model the dissipative properties of the material.

The terms Q_e^R, Q_e^θ, Q_e^f (i.e., the external actions) are computed on the basis of the forces and torques exchanged at the two connectors with the following code:

```
QeR=matrix(fa+fb_a);
QeTheta=matrix(ta+tb_a+cross(({L,0,0}
        +S1*B[N,:,:]*qf),fb_a));
Qef=transpose((transpose(matrix(fb_a))*S1*
B[N,:,:]+transpose(matrix(ta))*dS0*B[1,:,:])
+transpose(matrix(tb_a))*dS1*B[N,:,:]));
```

where `fa` and `fb_a` are the forces at the connectors, `ta` and `tb_a` the moments at the connectors, `S1` the matrix S_{el} evaluated for $\xi = 1$, `B[N,:,:]` the connectivity matrix B_N and `dS0` and `dS1` are matrices used to select the flanges moments acting on the deformation field; forces and moments are referred to the root flange coordinate system.

Figure 6: Cart with flexible inverted pendulum

The model parameters include the beam length and cross sectional area, the material density and Young modulus, the cross sectional inertia, the damping factor and the number of elements.

Particular care has been put into the realization of a 3D interface for the model to visualize the simulation results (Fig. 6), implemented by exploiting the features of the graphical environment of the multibody library. The 3D visualization has revealed itself to be an important feature, giving significative insight and sensible feedback about the dynamical behaviour of the model.

The finite volume model and the mixed one can be easily obtained by connecting several finite element beams composed by one or more elements, respectively. The achievement of such results, which significantly simplify the models implementation, is based on the modular approach adopted in the finite element model development. The assembly of the equations of motion for these cases is demanded to Modelica-based simulation environments, which usually employ advanced symbolic manipulation techniques and index reduction algorithms.

The dynamical properties of the latter models are significantly complex and accurate, featuring a displacement description which is fully non-linear and allowing the simulation of large displacement due to deformation (Fig. 7) at the cost, though, of a significant increase of the computational complexity with respect to the “pure” finite element model.

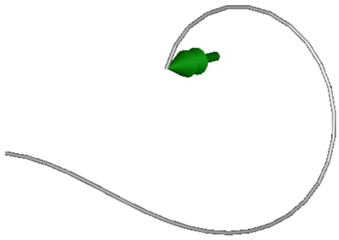

Figure 7: Large deformation of a thin beam

Mode	Freq.* [Hz]	Freq.† [Hz]	Error [%]
1	2.0854733	2.0854750	8.418e-005
2	13.0694381	13.0698705	3.308e-003
3	36.5948052	36.6041219	2.545e-002
4	71.7112127	71.7795490	9.529e-002
5	118.543772	118.842591	2.521e-001

* Theoretical prediction † Simulation result

Table 1: Theoretical and model natural frequencies

5 Simulations

The different flexible beam models have been validated by several simulation analysis performed within the Dymola simulation environment [1]. The most significative ones are reported in the following subsections.

5.1 Free Vibration

In this simulation the free vibration of a flexible beam is analyzed. The test-case has been set up in order to investigate the models properties with respect to theoretical predictions.
The beam component is connected to the world reference system, so that no rigid motion is allowed; furthermore, no gravity field is considered.
At the initial time instant the beam is standing still with a non-null tip displacement, then it evolves, vibrating, towards steady state.
The vibration frequencies of a flexible beam clamped at the root can be calculated by solving the following partial differential equation:

$$\rho\frac{\partial^2 y(x,t)}{\partial t^2}+EJ\frac{\partial^4 y(x,t)}{\partial x^4}=0 \tag{39}$$

with the following boundary and initial conditions:

$$\begin{cases} y(0,t),\ \dfrac{\partial y}{\partial x}(0,t),\ \dfrac{\partial^2 y}{\partial x^2}(0,t),\ \dfrac{\partial^3 y}{\partial x^3}(0,t)=0 \\ y(x,0)=f(x),\ \dfrac{\partial y}{\partial t}(x,0)=0 \end{cases} \tag{40}$$

where x is the axial coordinate, y is the transversal displacement and $f(x)$ is the initial deformation field. In [3] it is shown that the general solution for equation (39) has the following expression:

$$y(x,t)=\sum_{k=1}^{\infty}\varphi_k(x)\alpha_k(t)\,, \tag{41}$$

where $\varphi_k(x)$ are the spatial eigenfunctions and $\alpha_k(t)$ are periodical functions, with natural pulsation (corresponding to the k^{th} mode of vibration) given by:

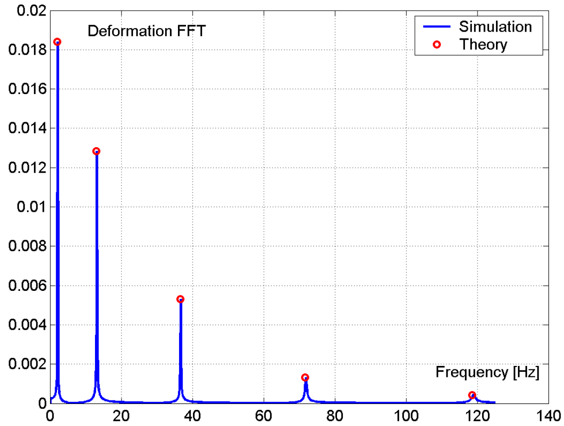

Figure 8: Tip displacement frequency spectrum

$$\omega_k=\beta_k^2\sqrt{\frac{EJ}{\rho}}, \tag{42}$$

being β_k the k^{th} root of the characteristic equation:

$$\cos(\beta L)\cosh(\beta L)+1=0 \tag{43}$$

The beam, made by aluminium, has square cross section $A=1\,cm^2$, length $L=2\,m$, density $\rho=2700\,kg/m^3$, Young's modulus $E=7.2\cdot10^9 N/m^2$ and has been discretized with $N=10$ elements. The initial tip displacement is $1\,cm$.
Table (1) contains a comparison between the results for for the first five vibrational modes obtained by simulation and by solving numerically equation (43). The results are in good accordance, as it is shown also in Fig. 8, depicting the tip displacement frequency spectrum.

5.2 Flexible Pendulum

This simulation, reported also in [2], involves the analysis of the vibrations induced by motion in a flexible pendulum swinging under the action of gravity.

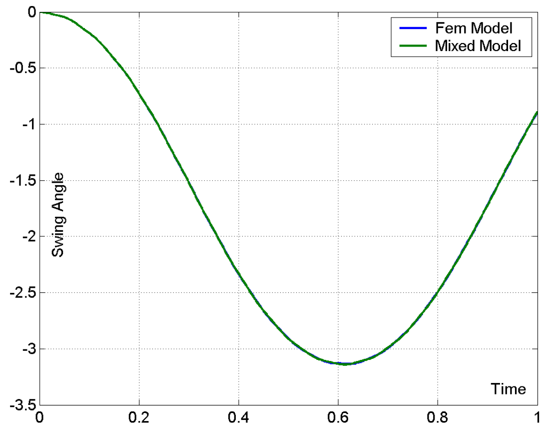

Figure 9: Swing angle

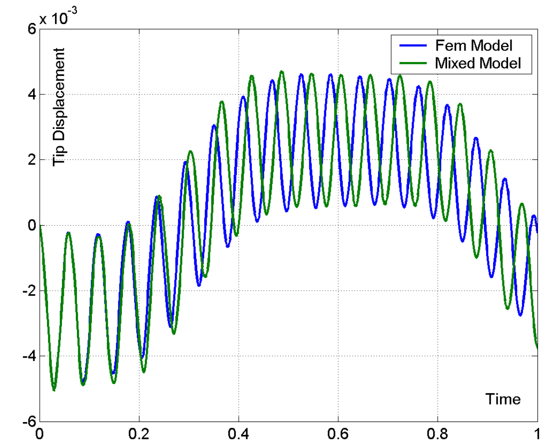

Figure 10: Tip displacement

The pendulum, connected to the world reference system by a revolute joint, has a length $L = 0.4\,m$, cross sectional area $A = 18\,cm^2$, density $\rho = 5540\,kg/m^3$, second moment of area $J = 1.215 \cdot 10^{-8}\,m^4$ and modulus of elasticity $E = 10^9\,N/m^2$. Two different models have been simulated: the first one composed by 10 elements and the second one by 5 volumes with 2 elements each.

In Fig. 9 the swing angle is depicted for both cases. The tip deformation, depicted in Fig. 10, appears to be slightly different for the two models. The results reported in [2] are in accordance with the ones obtained with the mixed model, though.

5.3 Elastic Slider-crank Mechanism

The simulation of an elastic slider-crank mechanism, reported also in [2], has been performed to validate the models for use within closed-loop mechanical chains.

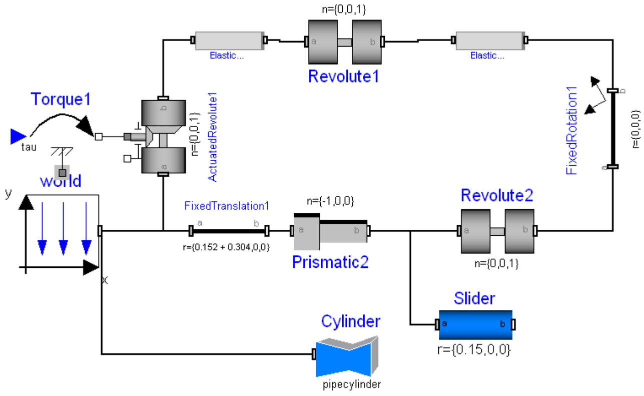

Figure 11: Slider-crank mechanism (Dymola scheme)

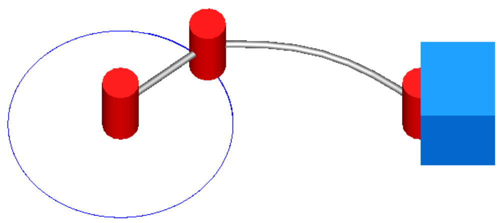

Figure 12: Slider-crank mechanism

The simulation set up involves a slider, a rod and a crankshaft connected by revolute joints (Fig. 11 and 12)

The crank has length $L = 0.152\,m$, cross sectional area $A = 0.7854\,cm^2$ and second moment of area $J = 4.909 \cdot 10^{-10}\,m^4$, density $\rho = 2770\,kg/m^3$ and modulus of elasticity $E = 10^9\,N/m^2$. The connecting rod has the same physical parameters of the crank, apart from the lenght $L = 0.304\,m$ and the Young's modulus $E = 5 \cdot 10^7 N/m^2$. The crank and the connecting rod have been discretized with 3 and 8 elements, respectively. Finally, the slider block has been assumed to be a massless rigid body.

During the simulation, the crankshaft is driven by a torque with the following law:

$$\begin{cases} M(t) = [0.01\,(1 - e^{-t/0.167})]Nm & , \quad t \leq 0.7\,sec \\ 0 & , \quad t > 0.7\,sec \end{cases} \qquad (44)$$

Fig. 13 and 14 show the slider position and the connecting rod tip transverse displacement, respectively. The results are in perfect accordance with those reported in [2].

6 Conclusion and Future Work

In this paper, a new model for flexible thin beams in *Modelica* is introduced. The model, fully compatible with the *MultiBody* library, is based on the application of the finite element method. Selected simulation results have been presented in order to validate the model properties with respect to scientific literature reference cases.

Future work will include the model extension to handle full 3D deformation and distributed loads. The model will also be employed for the development of applications in the field of robot control and satellite attitude control.

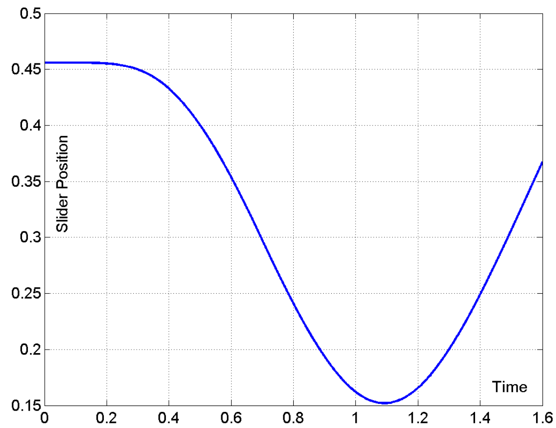

Figure 13: Slider block position

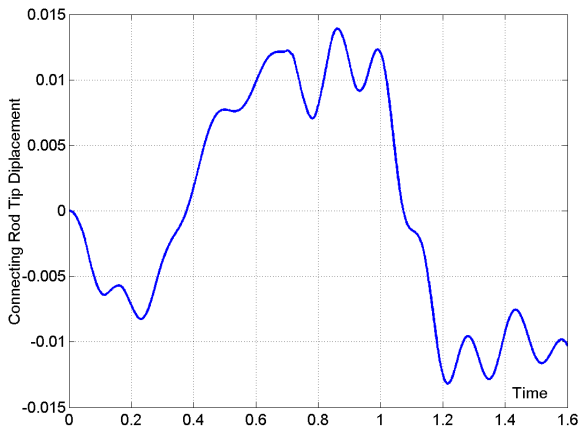

Figure 14: Transverse displacement of the tip of the connecting rod

A Structural Stiffness Matrix

$$K_{ff,el} = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ & \frac{12EJ}{3\ell} & \frac{6EJ}{3\ell} & 0 & -\frac{12EJ}{3\ell} & \frac{6EJ}{2\ell} \\ & & \frac{4EJ}{\ell} & 0 & -\frac{6EJ}{2\ell} & \frac{2EJ}{\ell} \\ & & & \frac{EA}{\ell} & 0 & 0 \\ & & & & \frac{12EJ}{3\ell} & -\frac{6EJ}{2\ell} \\ & & & & & \frac{4EJ}{\ell} \end{bmatrix}$$

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