Mathematical Aspects of Object-Oriented Modeling and Simulation

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Wien

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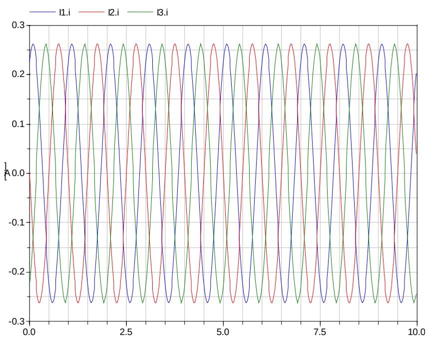
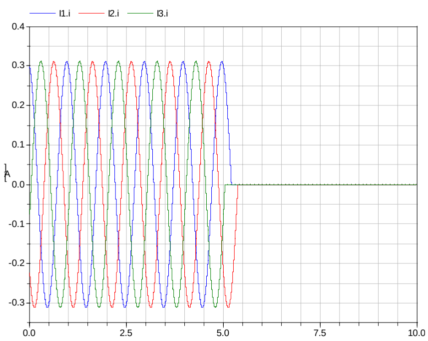
Outline

- Continuous system simulation
  - Introductory example
  - Symbolic transformation
  - Efficiency issues and non-linearities
  - Higher index problems
  - Initialization
  - Numerical issues

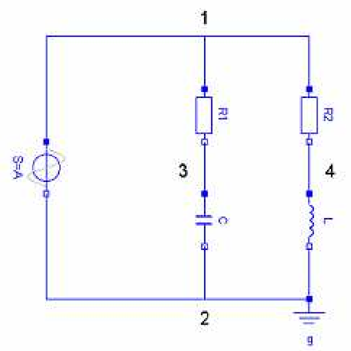


- Mixed system simulation
  - Event handling
  - State events ⇔ time events
  - Symbolic transformation
  - Ideal components
  - Varying higher index problems
  - Efficiency and numerical issues

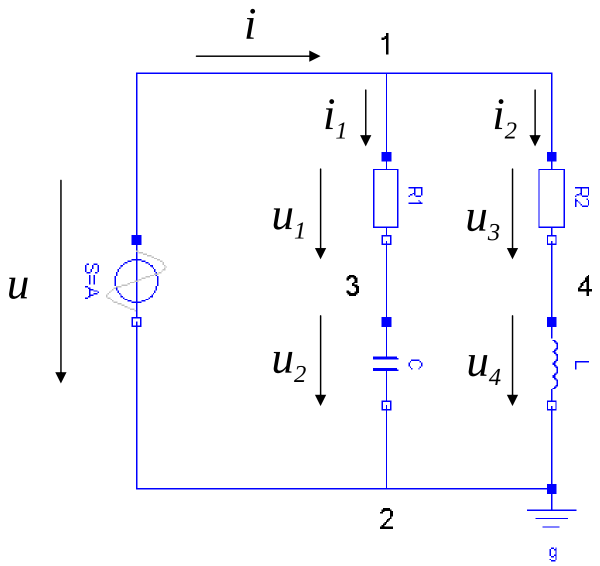


Introductory Example

- Flat model description
  - Differential equations, algebraic equations
  - Parameters, constants, and continuous variables
- Mathematical formalism
  - Differential-algebraic equations (DAEs)
  - Principles of numerical solution methods
  - Same number of equations and unknowns
  - Basic transformation steps
  - Explicit state-space representation
- Hierarchical modeling
  - Local description by equations
  - Connection equations describe interaction
  - Elimination of trivial equations
  - Explicit state-space representation



Example: A Simple Electrical System Flat Representation



$$u = A \cdot \sin(\omega\, t)$$

$$u = u_1 + u_2, \quad u = u_3 + u_4$$

$$i = i_1 + i_2$$

$$u_1 = R_1 i_1, \quad u_3 = R_2 i_2,$$

$$C \frac{du_2}{dt} = i_1, \quad L \frac{di_2}{dt} = u_4$$

The dynamical behavior of the system is given by a

System of differential-algebraic equations (DAEs)

Mathematical Formalism

General representation of DAEs:

$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

t	time
$\underline{\dot{x}}(t)$	vector of differentiated state variables
$\underline{x}(t)$	vector of state variables
$\underline{y}(t)$	vector of algebraic variables
$\underline{u}(\mathrm{t})$	vector of input variables
$\underline{p}$	vector of parameters and/or constants

Example: A Simple Electrical System

Model of the electrical system:

$$u = A \cdot \sin(\omega\, t + \varphi)$$
$$u = u_1 + u_2, \quad u = u_3 + u_4$$
$$i = i_1 + i_2$$
$$u_1 = R_1\ i_1, \quad u_3 = R_2\ i_2$$
$$C\ \dot{u}_2 = i_1, \quad L\ \dot{i}_2 = u_4$$

- 8 equations!
- Number of unknowns?
- Solution of the system?

General representation:

$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$t \ \text{ time}, \quad \underline{\dot{x}}(t) = \begin{pmatrix} \dot{u}_2 \\ \dot{i}_2 \end{pmatrix}, \quad \underline{x}(t) = \begin{pmatrix} u_2 \\ i_2 \end{pmatrix},$$

$$\underline{y}(t) = \begin{pmatrix} u \\ u_1 \\ u_3 \\ u_4 \\ i \\ i_1 \end{pmatrix}, \quad \underline{u}(\mathrm{t}) \text{ not present}, \quad \underline{p} = \begin{pmatrix} A \\ \omega \\ R_1 \\ R_2 \\ C \\ L \end{pmatrix}$$

Understand Numerical Integration Methods (Explicit Euler Method)

Integration of explicit ordinary differential equations (ODEs):

$$\underline{\dot{x}}(t) = \underline{f}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right), \qquad \underline{x}(t_0) = \underline{x}_0$$

Numerical approximation of the derivative and/or right-hand-side:

$$\underline{\dot{x}}(t_n) \approx \frac{\underline{x}(t_{n+1}) - \underline{x}(t_n)}{t_{n+1} - t_n} \approx \underline{f}\left(t_n, \underline{x}(t_n), \underline{u}(t_n), \underline{p}\right)$$

Iteration scheme:

$$\underline{x}(t_{n+1}) \approx \underline{x}(t_n) + \left(t_{n+1} - t_n\right) \cdot \underline{f}\left(t_n, \underline{x}(t_n), \underline{u}(t_n), \underline{p}\right)$$

Calculating an approximation of $\underline{x}(t_{n+1})$ based on the values of $\underline{x}(t_n)$

Here:
Explicit Euler integration method

Convergence?

Basic Transformation Steps

Transformation to explicit state-space representation:

$$0 = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$0 = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \qquad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\nearrow$$

$$\underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix} = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

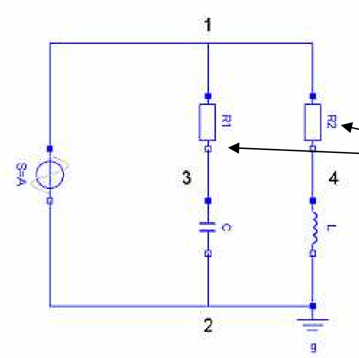
Implicit function theorem:
Necessary condition for the existence of the transformation is that the following matrix is regular at the point of interest:

$$\det\left(\frac{\partial}{\partial \underline{z}} \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right)\right) \neq 0$$

Example: A Simple Electrical System

Model of the electrical system:

$$u = A \cdot \sin(\omega\, t)$$
$$u = u_1 + u_2, \quad u = u_3 + u_4, \quad i = i_1 + i_2$$
$$u_1 = R_1 i_1, \quad u_3 = R_2 i_2$$
$$C\dot{u}_2 = i_1, \quad L\dot{i}_2 = u_4$$



Causality!

Transformation to explicit state space representation:

$$u = A \cdot \sin(\omega\, t)$$
$$\boxed{u_3 = R_2 i_2}$$
$$u_1 = u - u_2$$
$$u_4 = u - u_3$$
$$\boxed{i_1 = {}^{u_1}\!/_{R_1}}$$
$$i = i_1 + i_2$$
$$\dot{u}_2 = {}^{i_1}\!/_{C}$$
$$\dot{i}_2 = {}^{u_4}\!/_{L}$$

Transformed Simulation Model

Model of the electrical system in explicit state space representation:

$$u = A \cdot \sin(\omega\, t)$$
$$u_3 = R_2 i_2$$
$$u_1 = u - u_2$$
$$u_4 = u - u_3$$
$$i_1 = {}^{u_1}\!/_{R_1}$$
$$i = i_1 + i_2$$
$$\dot{u}_2 = {}^{i_1}\!/_{C}$$
$$\dot{i}_2 = {}^{u_4}\!/_{L}$$

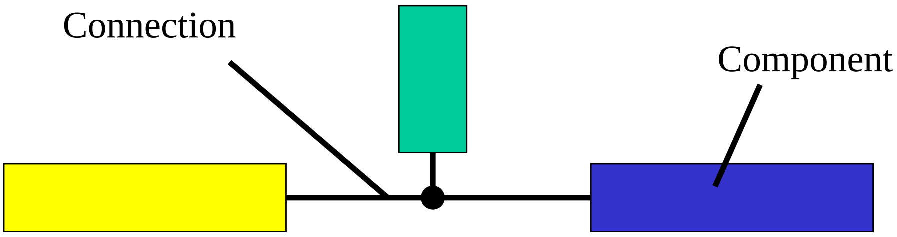
General representation:

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$
$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\begin{pmatrix} \dot{u}_2(t) \\ \dot{i}_2(t) \end{pmatrix} = \underline{h}\left(t, \underbrace{u_2(t), i_2(t)}_{\text{states}}, \underbrace{A, \omega, R_1, R_2, C, L}_{\text{parameters/constants}}\right)$$

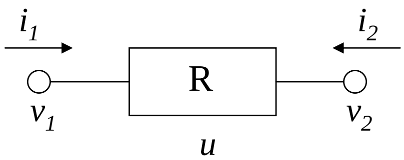
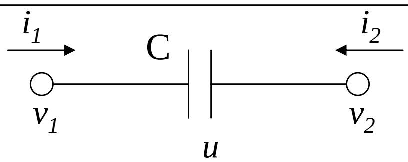
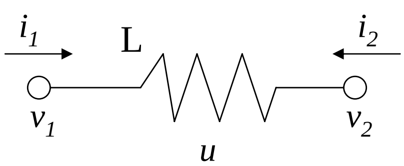
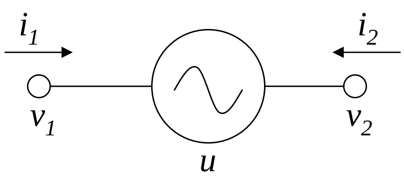
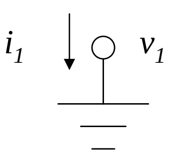
$$\begin{pmatrix} u(t) \\ u_1(t) \\ u_3(t) \\ u_4(t) \\ i(t) \\ i_1(t) \end{pmatrix} = \underline{k}\left(t, \underbrace{u_2(t), i_2(t)}_{\text{states}}, \underbrace{A, \omega, R_1, R_2, C, L}_{\text{parameters/constants}}\right)$$

Hierarchical Object-Oriented Modeling



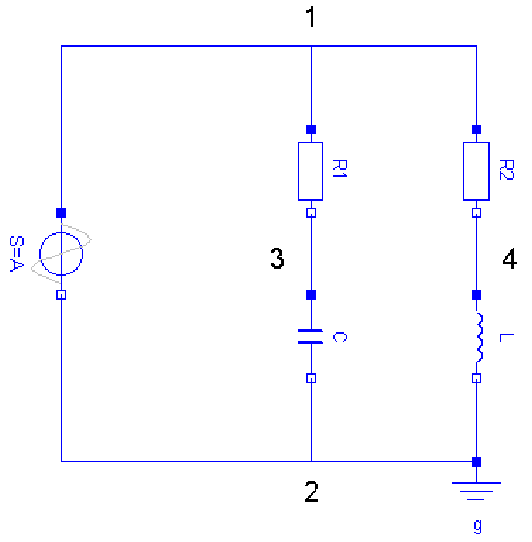
- Each icon represents a physical component
  - Electrical resistor, mechanical gearbox, pump
- Composition lines are the actual physical connections
  - Electrical line, mechanical connection, heat flow between two components
- Variables at the interfaces describe interaction with other components
- Physical behavior of a component is described by equations
- Hierarchical decomposition of components

Hierarchical Model of the Electrical System

Resistor		$0 = i_1 + i_2$ $u = v_1 - v_2$ $u = R\, i_1$
Capacitor		$0 = i_1 + i_2$ $u = v_1 - v_2$ $i_1 = C\, du/dt$
Inductor		$0 = i_1 + i_2$ $u = v_1 - v_2$ $u = L\, di_1/dt$
Voltage source		$0 = i_1 + i_2$ $u = v_1 - v_2$ $u = A \sin(wt)$
Ground		$v_1 = 0$

Hierarchical Model of the Electrical System

$0 = R1.i_1 + R1.i_2$ $R1.u = R1.v_1 - R1.v_2$ $R1.u = R1.R \cdot R1.i_1$	**R1**	$0 = R2.i_1 + R2.i_2$ $R2.u = R2.v_1 - R2.v_2$ $R2.u = R2.R \cdot R2.i_1$	**R2**
$0 = C.i_1 + C.i_2$ $C.u = C.v_1 - C.v_2$ $C.i_1 = C.C \cdot dC.u/dt$	**C**	$0 = L.i_1 + L.i_2$ $L.u = L.v_1 - L.v_2$ $L.u = L.L \cdot dL.i_1/dt$	**L**
$0 = S.i_1 + S.i_2$ $S.u = S.v_1 - S.v_2$ $S.u = S.A \sin(S.w \cdot t)$	**S**	$g.v_1 = 0$	**g**
$S.v_1 = R1.v_1$ $S.v_1 = R2.v_1$ $0 = S.i_1 + R1.i_1 + R2.i_1$	**1**	$R1.v_2 = C.v_1$ $0 = R1.i_2 + C.i_1$	**3**
$g.v_1 = C.v_2$ $g.v_1 = L.v_2$ $g.v_1 = S.v_2$ $0 = g.i_1 + C.i_2 + L.i_2 + S.i_2$	**2**	$R2.v_2 = L.v_1$ $0 = R2.i_2 + L.i_1$	**4**



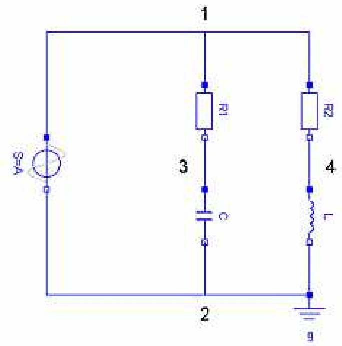
- 27 equations!
- number of unknowns?
- many trivial equations!

Hierarchical Model of the Electrical System

Transformation to explicit state space representation with elimination of trivial equations and further simplifications

$$
\begin{aligned}
R1.v_1 &:= S.A \cdot \sin(S.w \cdot t) \\
R2.u &:= R2.R \cdot L.i_1 \\
R1.u &:= R1.v_1 - C.u \\
L.u &:= R1.v_1 - R2.u \\
C.i_1 &:= R1.u / R1.R \\
S.i_1 &:= C.i_1 + L.i_1 \\
dC.u/dt &:= C.i_1 / C.C \\
dL.i_1/dt &:= L.u / L.L
\end{aligned}
$$

$\longleftrightarrow$



$$
\boxed{
\begin{aligned}
u &= A \cdot \sin(\omega\, t) \\
u_3 &= R_2 i_2 \\
u_1 &= u - u_2 \\
u_4 &= u - u_3 \\
i_1 &= u_1 / R_1 \\
i &= i_1 + i_2 \\
\dot{u}_2 &= i_1 / C \\
\dot{i}_2 &= u_4 / L
\end{aligned}
}
$$

Symbolic Transformation Algorithmic Steps

- DAEs and bipartite graph representation
  - Structural representation of the equation system
- The matching problem
  - Assign to each variable exact one equation
  - Same number of equations and unknowns
- Construct a directed graph
  - Find sinks, sources and strong components
  - Sorting the equation system
- Adjacence Matrix and structural regularity
  - Block-lower triangular form (BLT-Transformation)

$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{0} = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \quad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\downarrow$$

$$\underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix} = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

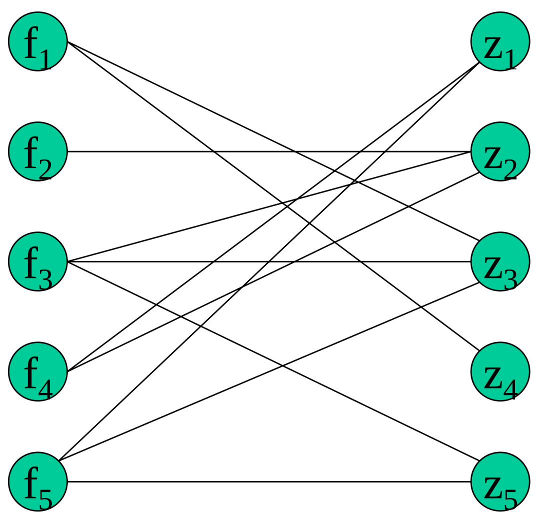
DAEs and Bipartite Graph Representation

Example of a regular DAE:

$$\underline{0} = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \quad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\begin{aligned} f_1(z_3, z_4) &= 0 \\ f_2(z_2) &= 0 \\ f_3(z_2, z_3, z_5) &= 0 \\ f_4(z_1, z_2) &= 0 \\ f_5(z_1, z_3, z_5) &= 0 \end{aligned}$$

Bipartite graph



Adjacence matrix

$$\begin{array}{c c} & \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

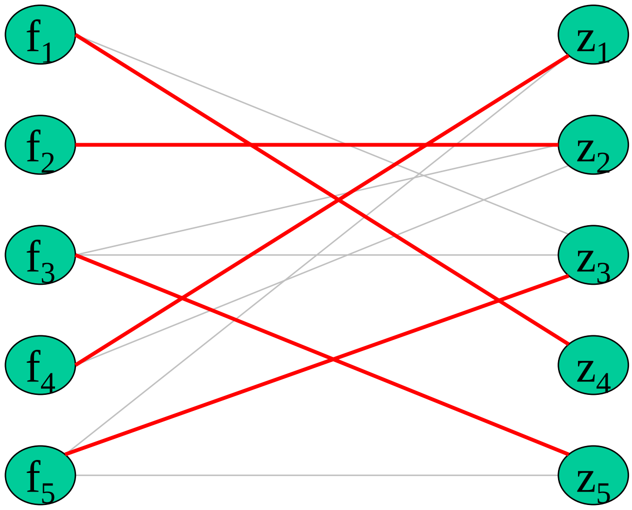
Solve the Matching Problem

Example of a regular DAE:

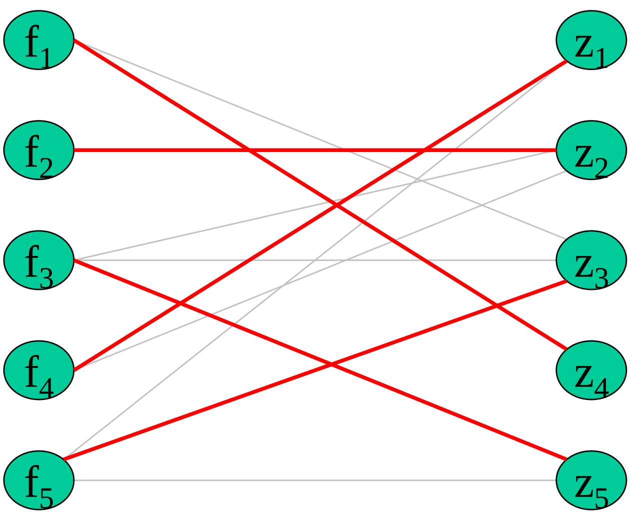
$$
\begin{aligned}
f_1(z_3,z_4) &= 0\\
f_2(z_2) &= 0\\
f_3(z_2,z_3,z_5) &= 0\\
f_4(z_1,z_2) &= 0\\
f_5(z_1,z_3,z_5) &= 0
\end{aligned}
$$

$$
\begin{array}{c c}
 & \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\
\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 1 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 1\\
1 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
$$

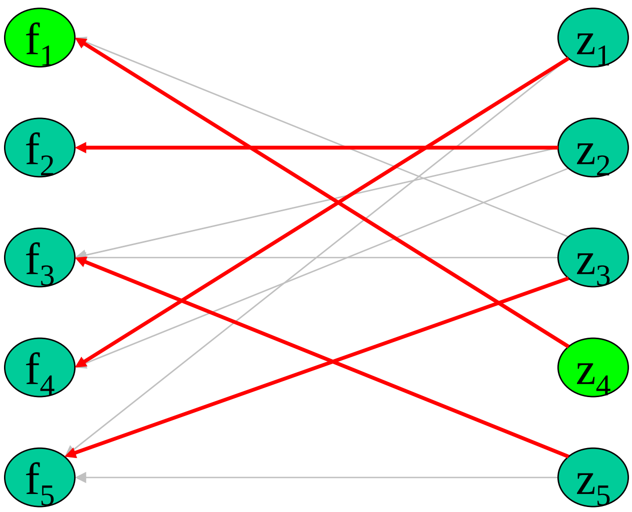
Bipartite graph



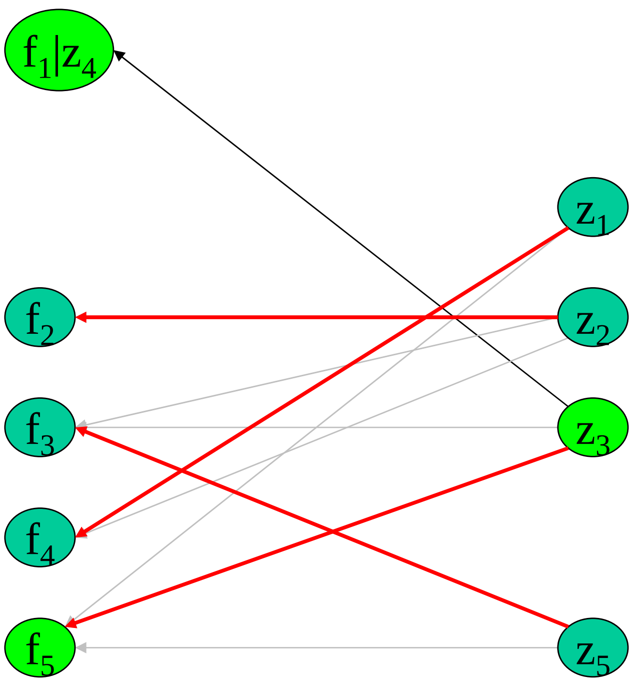
Construct a Directed Graph



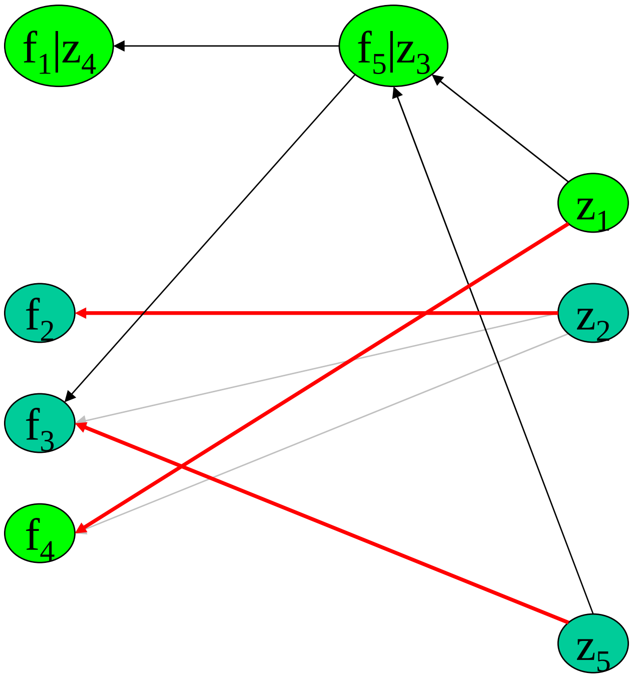
Construct a Directed Graph



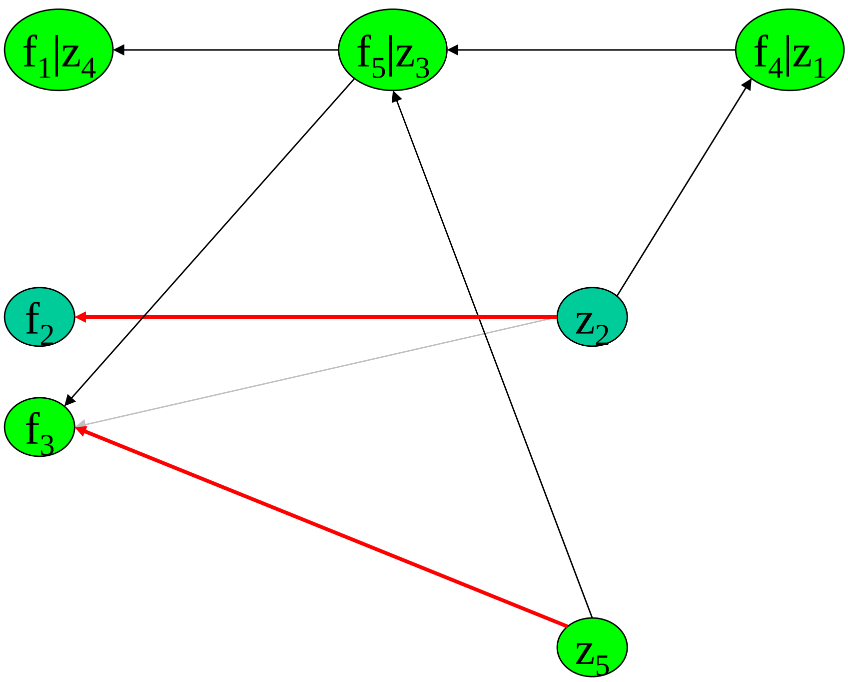
Construct a Directed Graph



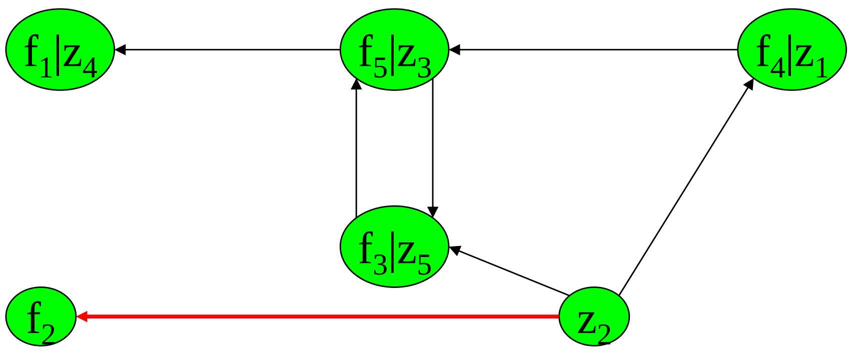
Construct a Directed Graph



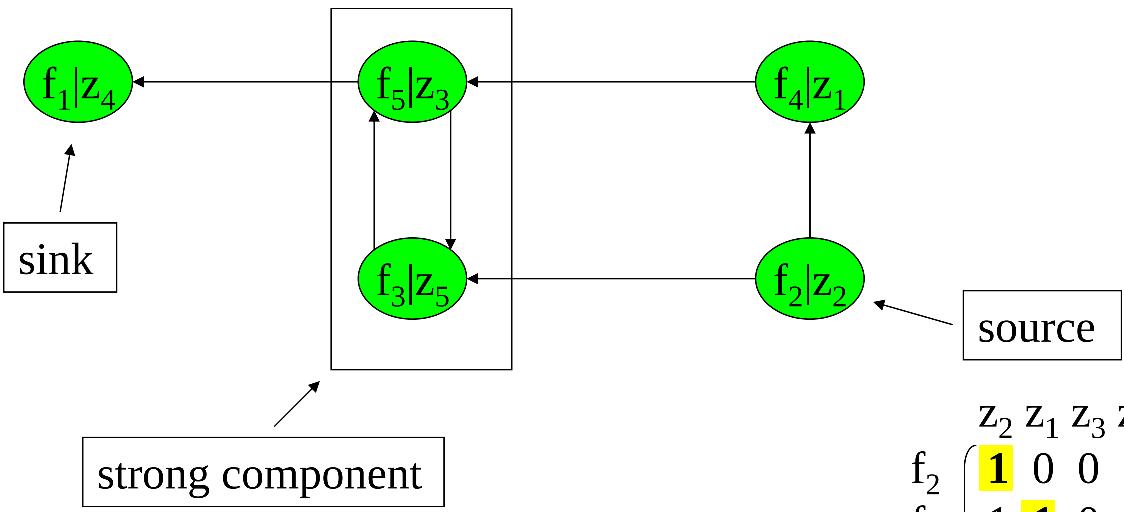
Construct a Directed Graph



Construct a Directed Graph



Construct a Directed Graph



$\Longrightarrow$ Tarjan's algorithm

$$
\begin{array}{c}
 \\ f_2 \\ f_4 \\ f_3 \\ f_5 \\ f_1
\end{array}
\begin{array}{c}
\begin{array}{ccccc} z_2 & z_1 & z_3 & z_5 & z_4 \end{array} \\
\begin{pmatrix}
\mathbf{1} & 0 & 0 & 0 & 0 \\
1 & \mathbf{1} & 0 & 0 & 0 \\
1 & 0 & \mathbf{1} & \mathbf{1} & 0 \\
0 & 1 & \mathbf{1} & \mathbf{1} & 0 \\
0 & 0 & 1 & 0 & \mathbf{1}
\end{pmatrix}
\end{array}
$$

Transformation Algorithms and the BLT-Form (Block-Lower-Triangle Form)

Example of a regular DAE:

$$
\begin{array}{lllll}
f_1(z_3,z_4) = 0 & \quad & f_1(z_3,[z_4]) = 0 & \quad & f_2(\underline{z}_2) = 0 \\
f_2(z_2) = 0 & & f_2([z_2]) = 0 & & f_4(\underline{z}_1,z_2) = 0 \\
f_3(z_2,z_3,z_5) = 0 & & f_3(z_2,z_3,[z_5]) = 0 & & f_3(z_2,\underline{z}_3,\underline{z}_5) = 0 \\
f_4(z_1,z_2) = 0 & & f_4([z_1],z_2) = 0 & & f_5(z_1,\underline{z}_3,\underline{z}_5) = 0 \\
f_5(z_1,z_3,z_5) = 0 & & f_5(z_1,[z_3],z_5) = 0 & & f_1(z_3,\underline{z}_4) = 0
\end{array}
$$

BLT-form of the adjacence matrix

$$
\begin{array}{c}
 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\
\begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\
\begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
 \\ f_2 \\ f_4 \\ f_3 \\ f_5 \\ f_1
\end{array}
\begin{array}{c}
\begin{array}{ccccc} z_2 & z_1 & z_3 & z_5 & z_4 \end{array} \\
\begin{pmatrix}
\mathbf{1} & 0 & 0 & 0 & 0 \\
1 & \mathbf{1} & 0 & 0 & 0 \\
1 & 0 & \mathbf{1} & \mathbf{1} & 0 \\
0 & 1 & \mathbf{1} & \mathbf{1} & 0 \\
0 & 0 & 1 & 0 & \mathbf{1}
\end{pmatrix}
\end{array}
$$

Structural Regularity of the Adjacence Matrix

- Transformation algorithm works on the model structure
  - Assign each variable to exact one equation (matching problem)
  - Find equations which have to be solved simultaneously
- Nevertheless, certain parameter settings could make a model singular

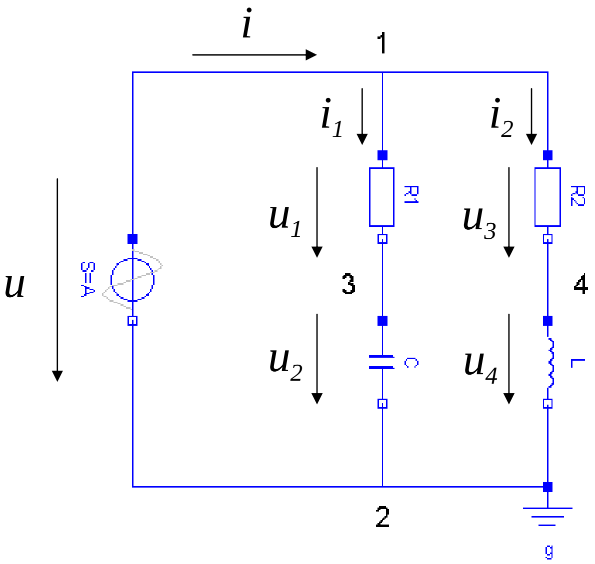
model singular

$$\Leftrightarrow \begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix} = a - b = 0$$

$$\Leftrightarrow a = b$$

```
model SingularModel
  parameter Real a=2;
  parameter Real b=1;
  Real x,y;
equation
  a*der(x) + b*der(y) = sin(time);
  der(x) + der(y) = cos(time);
end SingularModel
```

Example: A Simple Electrical System



$$\boxed{u} = A \cdot \sin(\omega\, t)$$

$$u = \boxed{u_1} + \boxed{u_2}$$

$$\boxed{u_3} = R_2 \boxed{i_2}$$

$$u = u_3 + \boxed{u_4}$$

$$u_1 = R_1 \boxed{i_1}$$

$$\boxed{i} = i_1 + i_2$$

$$C \frac{\boxed{du_2}}{dt} = i_1$$

$$L \frac{\boxed{di_2}}{dt} = u_4$$

Efficiency Issues and Non-Linear Equations

- Transformed equation system
  - Efficient calculation of the unknowns and derivatives of the states
  - Solve single linear equations based on causality
- Algebraic loops
  - Linear equation systems can be treated by direct or iterative routines
  - Non-linear equation systems need more sophisticated solution techniques (only local convergence!)
- Real-time aspects
  - Causality of non-linear equation may influence the runtime efficiency

$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{0} = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \quad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\downarrow$$

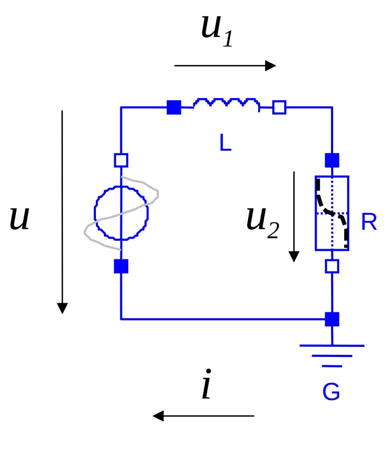
$$\underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix} = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

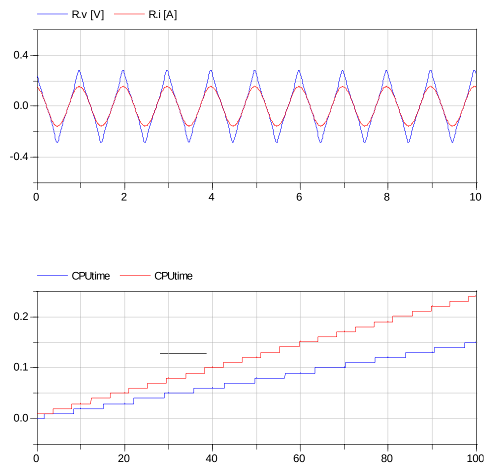
$$\downarrow$$

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

Example: A Non-Linear Electrical System





$$\boxed{u} = A \cdot \sin(\omega\ t)$$

$$i = a \tanh\left(b\ \boxed{u_2}\right)$$

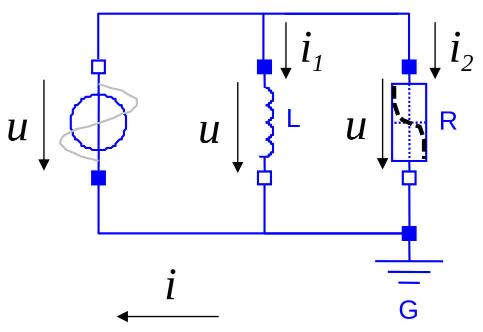
$$u = \boxed{u_1} + u_2$$

$$L \frac{\boxed{di}}{dt} = u_1$$

Re-writing the non-linear equation (Causality):

$$u_2 = \frac{1}{2b} \log\left({(1 + i/a)} \Big/ {(1 - i/a)} \right)$$

Example: A Non-Linear Electrical System



$$\boxed{u} = A \cdot \sin(\omega\ t)$$

$$\boxed{i_1} = a \tanh\left(b\ u\right)$$

$$L \frac{\boxed{di_2}}{dt} = u$$

$$\boxed{i} = i_1 + \boxed{i_2}$$

Causality of the non-linear equation correct

Higher Index Problems
Structurally Singular Systems

- **Higher-index DAEs**
  - Differential index of a DAE
  - Structural singularity of the adjacence matrix
  - Index reduction method using symbolic differentiation of equations
- **Numerical issues**
  - Consistent initialization
  - Drift phenomenon
  - Dummy derivative method
- **State selection mechanism in Modelica**
  - See also initialization of 3-phase electrical system

$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{0} = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \quad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\downarrow$$

$$\underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix} = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

Higher-Index-DAEs

General representation of DAEs:

$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

t	time
$\underline{\dot{x}}(t)$	vector of differentiated state variables
$\underline{x}(t)$	vector of state variables
$\underline{y}(t)$	vector of algebraic variables
$\underline{u}(\mathrm{t})$	vector of input variables
$\underline{p}$	vector of parameters and/or constants

Differential index of a DAE:

The minimal number of analytical differentiations of the equation system necessary to extract by algebraic manipulations an explicit ODE for all unknowns.

Higher-Index-DAEs

DAE with differential index 0:

$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right) \longrightarrow \underline{\dot{x}}(t) = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

DAE with differential index 1:

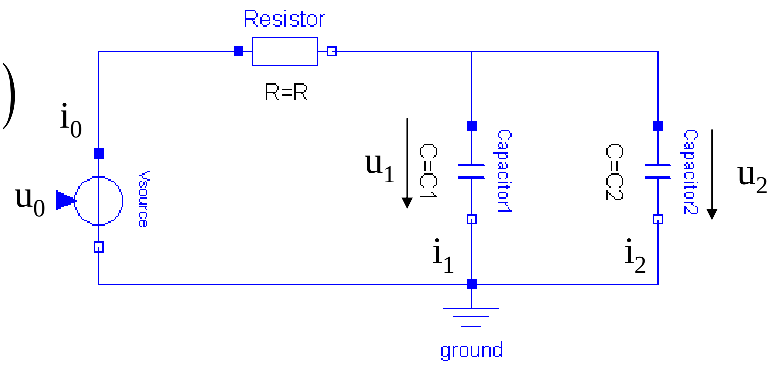
$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\begin{aligned} \underline{\dot{x}}(t) &= \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right) \\ \underline{y}(t) &= \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right) \end{aligned} \longrightarrow \begin{aligned} \underline{\dot{x}}(t) &= \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right) \\ \underline{\dot{y}}(t) &= \frac{d}{dt}\underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right) \end{aligned}$$

Higher-Index-DAE Example (index 2)

$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$



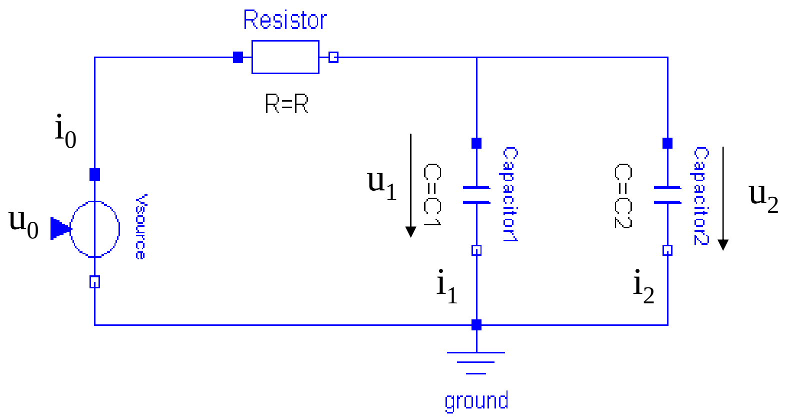
$$\downarrow$$

$$\begin{aligned} C_1 \cdot \dot{u}_1 &= i_1 \\ C_2 \cdot \dot{u}_2 &= i_2 \\ \boxed{u_1 = u_2} \\ i_0 &= i_1 + i_2 \\ u_0 &= R \cdot i_0 + u_1 \end{aligned} \qquad \mathbf{x} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ i_0 \\ i_1 \\ i_2 \end{bmatrix}$$

5 equations and 5 unknowns.
BUT: system is singular!

constrained equation between states

Higher-Index-DAE Example (index 2)



$$C_1 \cdot \dot{u}_1 = i_1$$
$$C_2 \cdot \dot{u}_2 = i_2$$
$$\boxed{\cancel{u_1 = u_2}}$$
$$i_0 = i_1 + i_2$$
$$u_0 = R \cdot i_0 + u_1$$
$$\dot{u}_1 = \dot{u}_2$$

Algorithmic steps
- Differentiation of constrained equations
- Extracting underlying ODE (DAE of index 1)

$$\mathbf{x} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ i_0 \\ i_1 \\ i_2 \end{bmatrix}$$

$\Longrightarrow$ ODE with 5 equations and 5 unknowns
- Consistent initialization necessary

Higher-Index-DAEs -- Numerical Problems

- Consistent initial conditions
  – Relation between states are eliminated when differentiating
  – Initial conditions need to be determined using the algebraic constrains
  – Automatic procedure possible using assign algorithm on the constrained equations
- Drift phenomenon
  – Algebraic constrained no longer fulfilled during simulation
  – Even worse when simulating stiff problems

=> Dummy-Derivative method

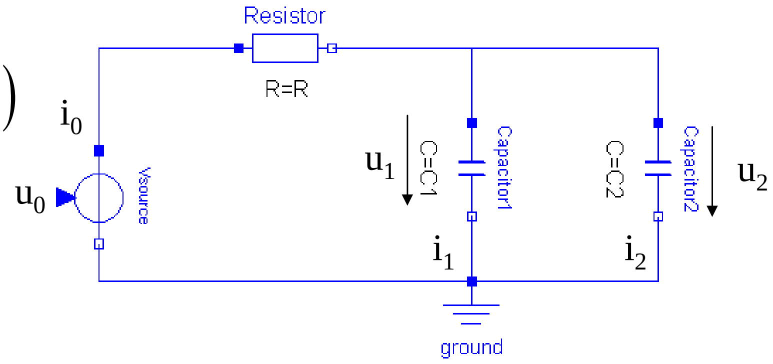
Principles of the Dummy-Derivative Method

- Matching algorithm fails
  - System is structurally singular
  - Find minimal subset of equations
    - more equations than unknown variables
  - Singularity is due to equations constraining states

- Differentiate subset of equations
  - Static state selection during compile time
    - choose one state and corresponding derivative as purely algebraic variable
      - so-called dummy-state and dummy derivative
    - by differentiation introduced variables are algebraic
    - continue matching algorithm
    - check initial conditions
  - Dynamic state selection during simulation time
    - store information on constrained states
    - make selection dynamically based on heuristic criteria
    - new state selection triggers an event (re-initialize states)

Dummy-Derivative method
Example (index 2)

$$\underline{0} = \underline{f}\left(t,\ \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$



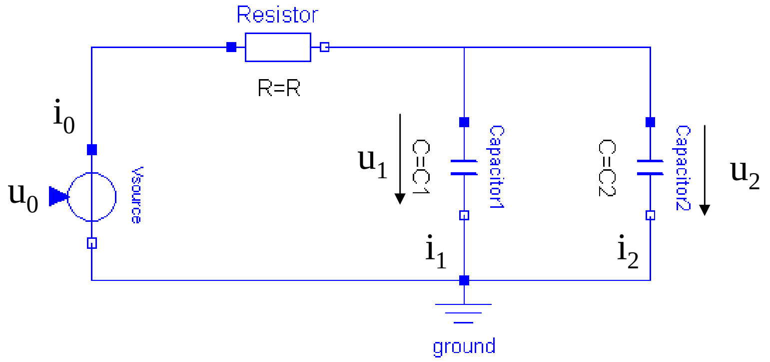
$$\downarrow$$

$$C_1 \cdot \dot{u}_1 = i_1$$
$$C_2 \cdot \dot{u}_2 = i_2$$
$$\boxed{u_1 = u_2}$$
$$i_0 = i_1 + i_2$$
$$u_0 = R \cdot i_0 + u_1$$

$$\mathbf{x} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ i_0 \\ i_1 \\ i_2 \end{bmatrix}$$

constrained equation

Dummy-Derivative method Example (index 2)



$$C_1 \cdot \dot{u}_1 = i_1$$
$$C_2 \cdot u_2' = i_2$$
$$u_1 = u_2$$
$$i_0 = i_1 + i_2$$
$$u_0 = R \cdot i_0 + u_1$$
$$\dot{u}_1 = u_2'$$

Algorithmic steps

- Differentiation of constrained equations
- Choose u_2 as dummy-state and u'_2 as dummy-derivative
- Extracting underlying ODE (DAE of index 1)

$\Longrightarrow$ ODE with 6 equations and 6 unknowns

- Check initial condition

$$\mathbf{x} = [u_1] \qquad \mathbf{y} = \begin{bmatrix} i_0 \\ i_1 \\ i_2 \\ u_2 \\ u_2' \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \dot{u}_1 \\ i_0 \\ i_1 \\ i_2 \\ u_2 \\ u_2' \end{bmatrix}$$

State Selection in Modelica

- Influence default state selection mechanism
  - Attribute `StateSelect stateSelect = StateSelect.default;`
  - Possible choices:

`never` - Do not use as state at all

`avoid` - Use as state, if it cannot be avoided (but only if variable appears differentiated and no other potential state with attribute default, prefer, or always can be selected)

`default` - Use as state if appropriate, but only if variable appears differentiated

`prefer` - Prefer it as state over those having the default value (also variables can be selected, which do not appear differentiated)

`always` - Do use it as a state

Initialization of Dynamic Models

- **Initialization of “free” state variables**
  - Transformed DAE after index-reduction
  - States can be chosen at start time
    - same number of additional equations and “free” states
- **Initialization of parameters**
  - Determine parameter settings
  - Parameters can be calculated at start time
    - same number of additional equations and “free” parameters
- **Initialization mechanism in Modelica**
  - attribute start
  - initial equation section
    - attribute fixed for parameters

$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{0} = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \quad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\downarrow$$

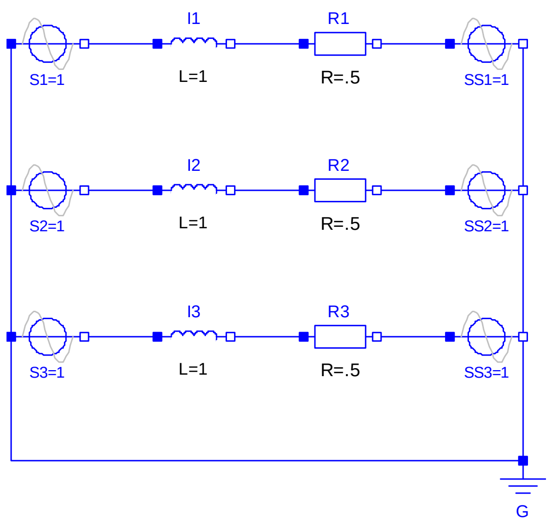
$$\underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix} = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

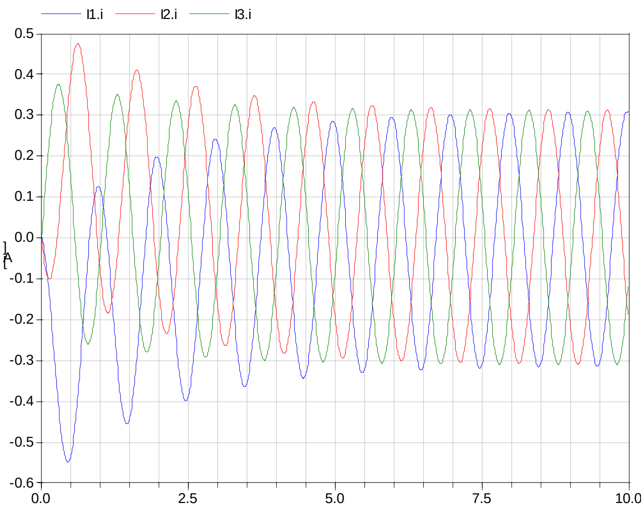
$$\downarrow$$

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

Example: 3-Phase Electrical System





Steady-state initialization?

- States `I1.i, I2.i, I3.i` are not constant at initial time!
- Here: Initial values set to 0 (standard settings, attribute `start`)
- Transformation to rotating reference system necessary

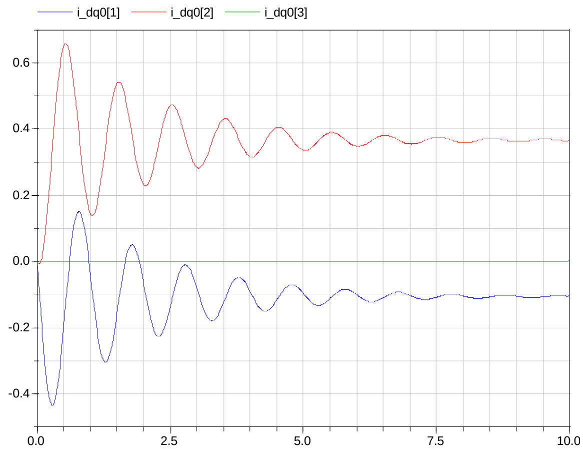
Example: 3-Phase Electrical System

Park-Transformation to rotating reference system:

- Write states as vector `i_abc[3] = {I1.i, I2.i, I3.i}`
- Park-transformation to dq0-reference frame

$$P = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} \sin(\omega t) & \sin\left(\omega t + \frac{2\pi}{3}\right) & \sin\left(\omega t + \frac{4\pi}{3}\right) \\ \cos(\omega t) & \cos\left(\omega t + \frac{2\pi}{3}\right) & \cos\left(\omega t + \frac{4\pi}{3}\right) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$i\_dq0 = P \cdot i\_abc$$

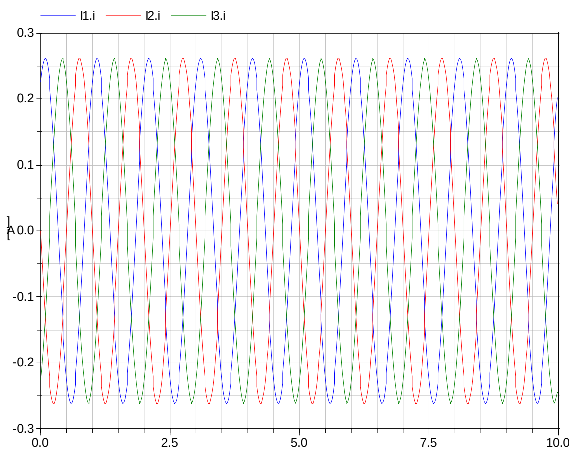


Example: 3-Phase Electrical System
Initialize States

```
model Test3PhaseSystem
  parameter Real shift=0.4;
  Real i_abc[3]={I1.i,I2.i,I3.i};
  Real i_dq0[3];
  …
initial equation
  der(i_dq0)={0,0,0};
equation
  …
  i_dq0 = P*i_abc;
end Test3PhaseSystem
```

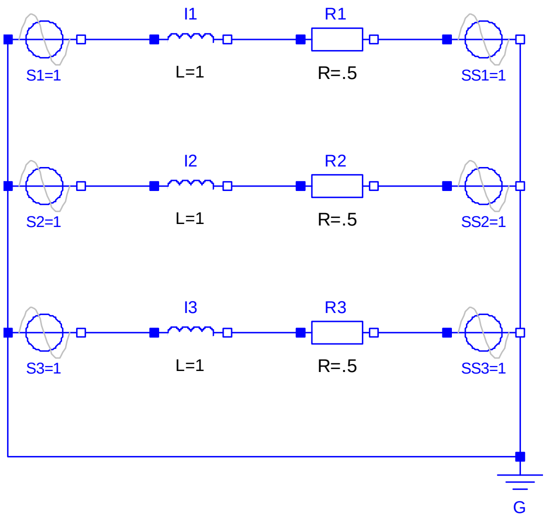
Steady-state initialization:

- Derivatives of `i_dq0` are only introduced during initialization
- Differentiation of `i_dq0 = P*i_abc` necessary
- No higher-index problem



Example: 3-Phase Electrical System
Initialize Parameters

```
model Test3PhaseSystem
  parameter Real shift(fixed=false,start=0.1);
  Real i_abc[3]={I1.i,I2.i,I3.i}, u_abc[3]={S1.v,S2.v,S3.v};
  …
initial equation
  der(i_dq0)={0,0,0};
  power = -0.12865;
equation
  …
  u_dq0 = P*u_abc;
  i_dq0 = P*i_abc;
  power = u_dq0*i_dq0;
end Test3PhaseSystem
```

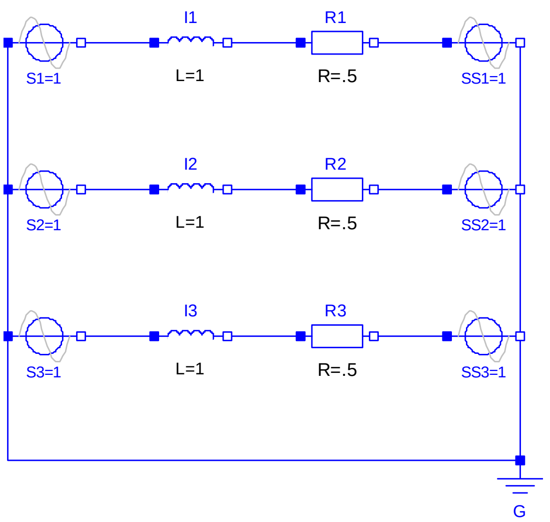


Parameter initialization:

- Non-linear equation, no unique solution
  `power=u_dq0*i_dq0=-0.12865`
- Attribute `fixed` (default `true`)
- Attribute `start` (default `0`)

Example: 3-Phase Electrical System
Influence State Selection

```
model Test3PhaseSystem
  Real i_abc[3]={I1.i,I2.i,I3.i};
  Real i_dq0[3](each stateSelect=StateSelect.always);
  …
initial equation
  …
equation
  …
  i_dq0 = P*i_abc;
end Test3PhaseSystem
```



State selection:

- Introduce `i_dq0` as states
  ```
  Real di_dq0[3];
  der(i_dq0)=di_dq0;
  ```
- Constrained equation between states must be differentiated
  `i_dq0 = P*i_abc;`

Example: 3-Phase Electrical System
Introduce Dummy-States/Derivatives

Mathematical formulas to the Park-transformation: (using trigonometric identities)

$$P = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} \sin(\omega t) & \sin\left(\omega t + \frac{2\pi}{3}\right) & \sin\left(\omega t + \frac{4\pi}{3}\right) \\ \cos(\omega t) & \cos\left(\omega t + \frac{2\pi}{3}\right) & \cos\left(\omega t + \frac{4\pi}{3}\right) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$i\_dq0 = P \cdot i\_abc$$

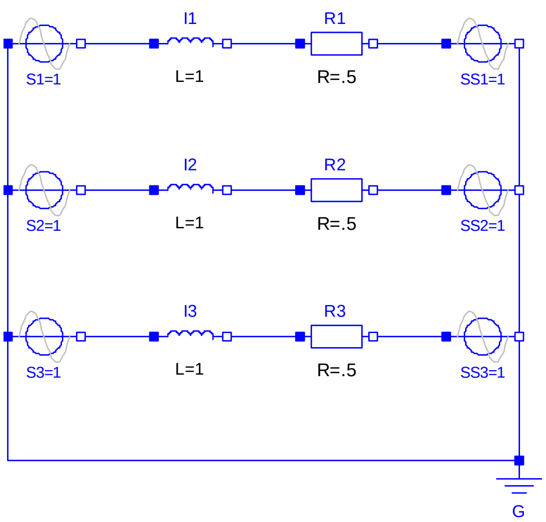
$$\downarrow$$

$$P \cdot P^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P' \cdot P^T = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\frac{d}{dt} i\_dq0 = \omega \cdot \begin{pmatrix} i\_dq0_2 \\ -i\_dq0_1 \\ 0 \end{pmatrix} + P \cdot \frac{d}{dt} i\_abc}$$

Example: 3-Phase Electrical System
Introduce Dummy-States/Derivatives

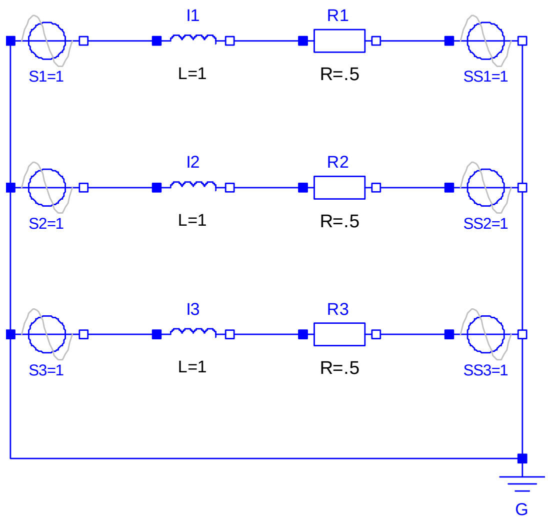
```
model Test3PhaseSystem
  Real i_abc[3]={I1.i,I2.i,I3.i};
  Real i_dq0[3];
  Real di_dq0[3]=2*pi*{i_dq0[2],-i_dq0[1],0} + P*der(i_abc);
  …
initial equation
  di_dq0={0,0,0};
equation
  …
  i_dq0 = P*i_abc;
end Test3PhaseSystem
```



State selection:

- Introduce `di_dq0` as dummy derivative
- Only used during initialization
- No differentiation necessary
- States `I1.i, I2.i, I3.i` used during simulation

Example: 3-Phase Electrical System
Introduce Dummy-States/Derivatives



State selection:

- Introduce `di_abc` as dummy derivative
- Rewrite inductor equation: `L*di = v;`
- No differentiation necessary
- Efficiency increase during simulation (real-time aspect)

```
model Test3PhaseSystem
  Real i_abc[3]={I1.i,I2.i,I3.i};
  Real di_abc[3]={I1.di,I2.di,I3.di};
  Real i_dq0[3];
  …
initial equation
  der(i_dq0)={0,0,0};
equation
  …
  i_dq0 = P*i_abc;
  der(i_dq0)=2*pi*{i_dq0[2],-i_dq0[1],0} + P*di_abc;
end Test3PhaseSystem
```


Numerical Integration of ODEs/DAEs

- Integration methods for ODEs
  - Single-step methods
    - explicit and implicit Runge-Kutta formulas
    - fixed and variable step-size
      - Richardson-Extrapolation
      - embedded Runge-Kutta Formulas (Dopri5)
  - Multi-step methods
    - explicit and implicit schemes
      - Adams/Bashforth/Moulton formulas
      - BDF-formulas (i.e. DASSL)
    - start-up using single-step methods (events)
  - Event handling
    - stop simulation and restart integration
- Stability region and stiffness detection
  - Linearize the system
  - Determine eigenvalues

$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{0} = \underline{f}\left(t, \underline{z}(t), \underline{x}(t), \underline{u}(t), \underline{p}\right), \quad \underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix}$$

$$\downarrow$$

$$\underline{z}(t) = \begin{pmatrix} \underline{\dot{x}}(t) \\ \underline{y}(t) \end{pmatrix} = \underline{g}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\downarrow$$

$$\underline{\dot{x}}(t) = \underline{h}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

$$\underline{y}(t) = \underline{k}\left(t, \underline{x}(t), \underline{u}(t), \underline{p}\right)$$

Stability Analysis of Runge-Kutta Methods

ODE problem: $\dot{\underline{x}}(t) = \underline{f}\left(t, \underline{x}(t)\right)$

describing disturbance: $\underline{y}'(t) = J(t) \cdot \underline{y}(t), \quad J(t) = \frac{\partial \underline{f}}{\partial \underline{x}}(t, \underline{x})$

Euler integration:
(J constant)

$$\underline{y}_{m+1} = \underline{y}_m + \Delta t \cdot J \cdot \underline{y}_m = R(\Delta t \cdot J) \cdot \underline{y}_m$$
$$R(z) = 1 + z$$

eigenvector $\underline{y}_0$
eigenvalue λ

$$\underline{y}_m = \left[R(\Delta t \cdot \lambda)\right]^m \cdot \underline{y}_0$$

bounded solution, iff $\left|R(\Delta t \cdot \lambda)\right| \le 1, \quad \lambda$ eigenvalue of J

Stability Analysis of Runge-Kutta Methods

Dahlquist test equation: $y' = \lambda \cdot y, \quad y_0 = 1, \quad z = \Delta t \cdot \lambda$

Stability function $R(z)$ is corresponding numerical solution at $t + \Delta t$:

Runge-Kutta method
(order=p):

$$R(z) = 1 + \frac{z}{1!} + \ldots + \frac{z^p}{p!} + O\left(z^{p+1}\right)$$

Runge-Kutta method
(order=s, explicit, s_stage):

$$R(z) = 1 + \frac{z}{1!} + \ldots + \frac{z^s}{s!}$$

Dopri5
(order=5, explicit, 6_stage):

$$R(z) = 1 + \frac{z}{1!} + \ldots + \frac{z^5}{5!} + \frac{z^6}{600}$$

bounded solution, iff $\left|R(\Delta t \cdot \lambda)\right| \le 1, \quad \lambda$ eigenvalue of J

Stability Region for Explicit Runge-Kutta Methods

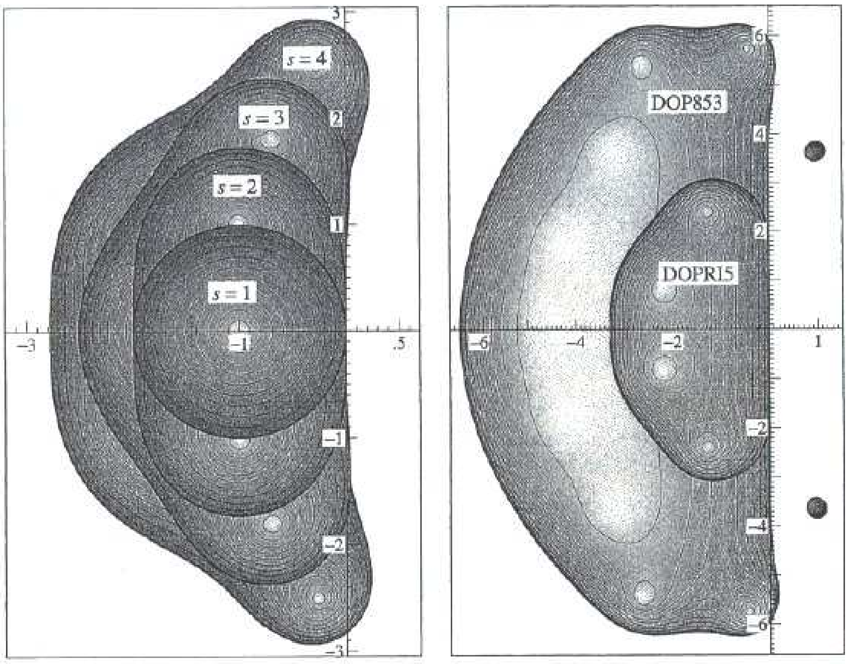


Fig. 2.1. Stability domains for explicit Runge-Kutta methods of order $p = s$

Fig. 2.2. Stability domains for DOPRI methods

Stability Region for Implicit Runge-Kutta Methods

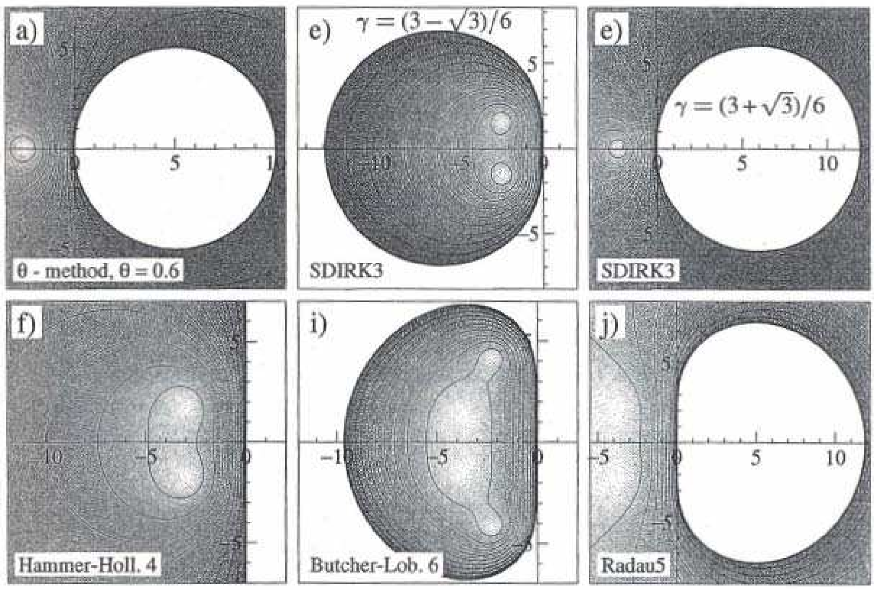


Fig. 3.1. Stability domains for implicit Runge-Kutta methods

Stability Region for Adams-Multi-Step Methods

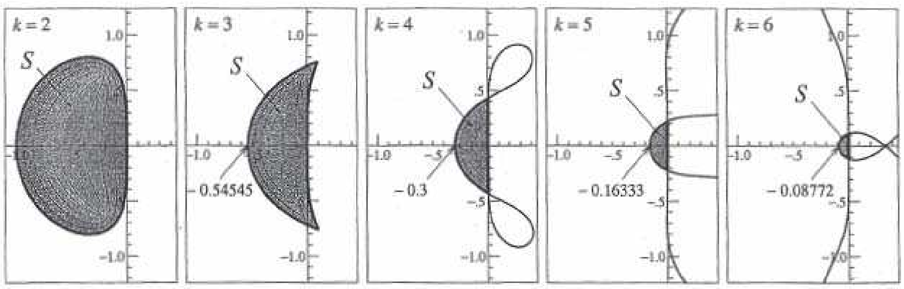


Fig. 1.2. Stability domains for explicit Adams methods

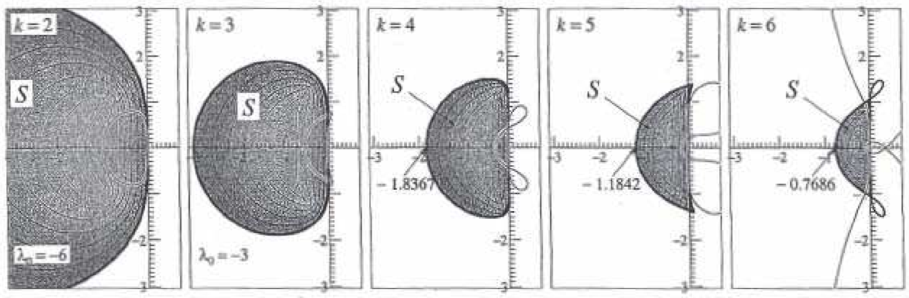


Fig. 1.3. Stability domains of implicit Adams methods, compared to those of the explicit ones

Stability Region for BDF – Methods (i.e. DASSL)

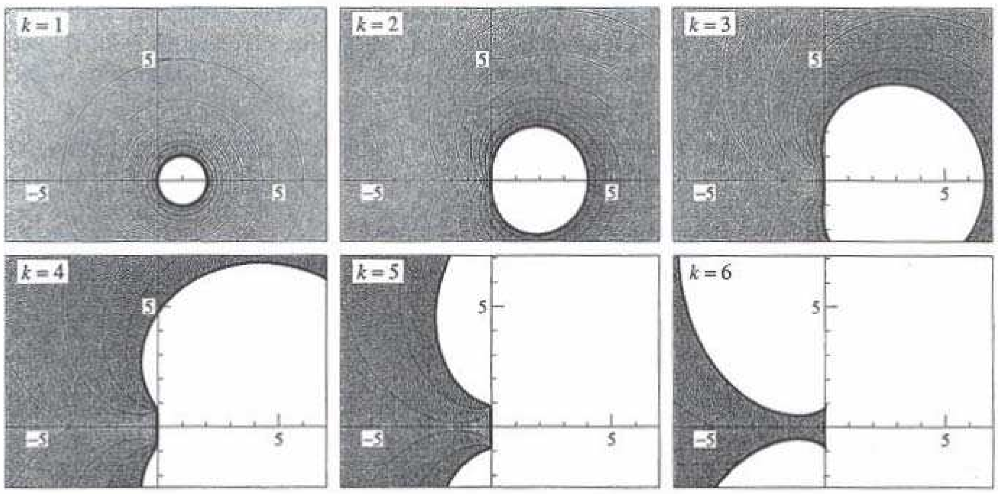


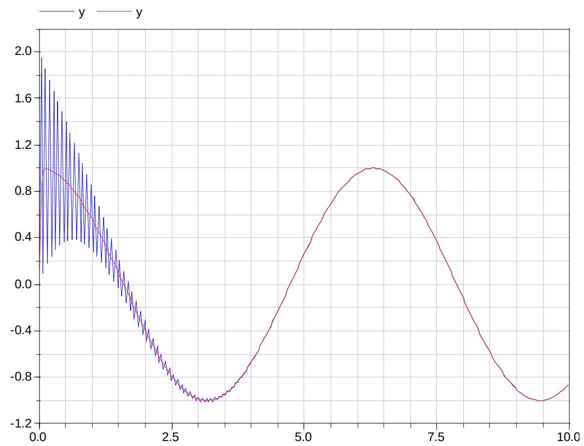
Fig. 1.6. Root locus curves and stability domains of BDF methods

Numerical Examples (Stability Region)

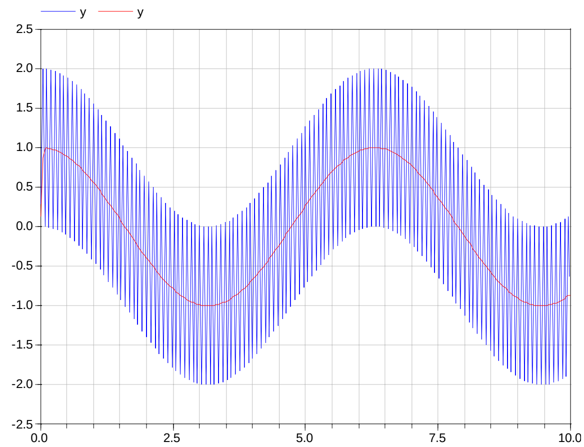
- Curtiss & Hirschfelder
  - Transition phase at the initial point
  - Determine maximal fixed step-size

$$y' = -50\left(y - \cos(t)\right), \quad \text{eigenvalue } \lambda = -50$$

$\Delta t = 0.039$: (explicit Euler)



$\Delta t = 0.04$: (explicit Euler)

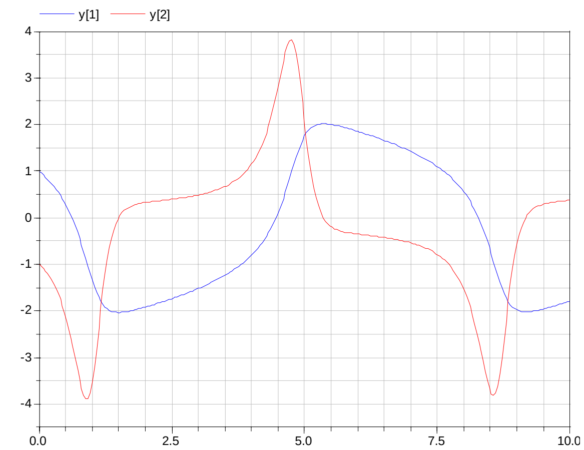


Numerical Examples (Stability Region and Stiffness)

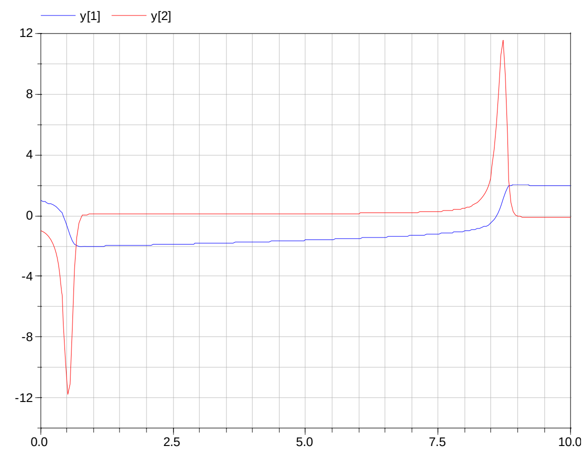
- Van der Pol
  - Simple nonlinear equation describing electrical systems
  - Stiff system, if $\varepsilon >> 1$.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \varepsilon\left(1 - y_1^2\right) y_2 - y_1 \end{aligned} \qquad \text{mit } J(t) = \begin{pmatrix} 0 & 1 \\ -2\varepsilon y_1 y_2 - 1 & \varepsilon\left(1 - y_1^2\right) \end{pmatrix}$$

$\varepsilon = 2$:



$\varepsilon = 8$:



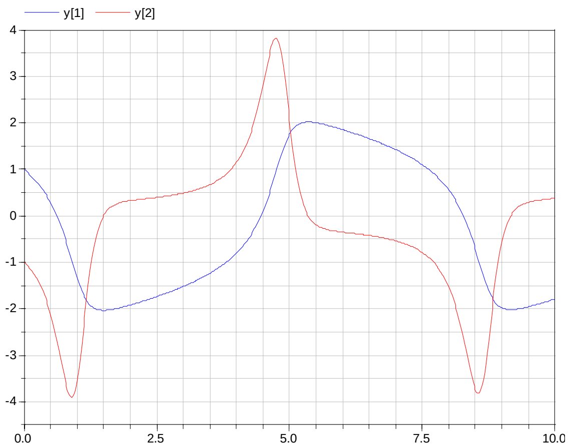
Determine Stability Region (Van der Pol equation)

- Linearize the system at initial time
  - Calculate the Jacobians
  - Read matrix into Matlab
  - Determine the eigenvalues
  - Calculate maximal step-size

$$\begin{aligned}\dot{\underline{x}}(t) &= \underline{h}\left(t,\underline{x}(t),\underline{u}(t),\underline{p}\right)\\ \underline{y}(t) &= \underline{k}\left(t,\underline{x}(t),\underline{u}(t),\underline{p}\right)\end{aligned} \longrightarrow \begin{aligned}\dot{\underline{x}}(t) &= A\underline{x}(t)+B\underline{u}(t)\\ \underline{y}(t) &= C\underline{x}(t)+D\underline{u}(t)\end{aligned}$$

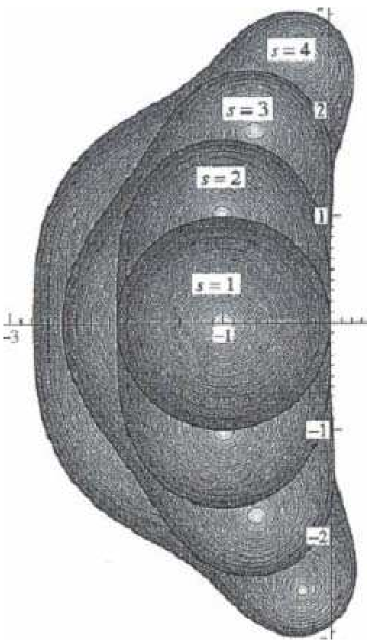
```
>> A = tloadlin
>c:\…\dslin.mat  loaded.
A =         0     1.0000
    39.0000          0
>> l=eig(A)
l =   6.245
     -6.245

>>delta_t=2/6.245
delta_t = 0.3203
```



Determine Stability Region (Van der Pol equation)

bounded solution, iff $\left|R(\Delta t\cdot\lambda)\right|\le 1,\quad \lambda$ eigenvalue of J



explicit Euler method: $R(z)=1+z$

$$\left|R(\Delta t\cdot\lambda)\right| = \left|1+\Delta t\cdot\lambda\right| = \left|\begin{pmatrix}1\\0\end{pmatrix}+\Delta t\begin{pmatrix}x\\y\end{pmatrix}\right|\le 1$$

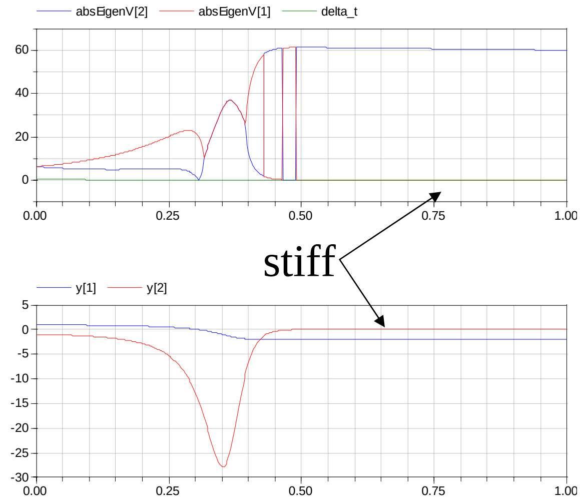
$$\Leftrightarrow (1+\Delta t\cdot x)^2+(\Delta t\cdot y)^2\le 1$$

$$\Leftrightarrow \Delta t^2\left(x^2+y^2\right)+2\cdot\Delta t\cdot x\le 0$$

$$\Leftrightarrow \Delta t\le -\frac{2\cdot x}{x^2+y^2}$$

Determine Stability Region (Van der Pol equation)

- Calculate eigenvalues during simulation

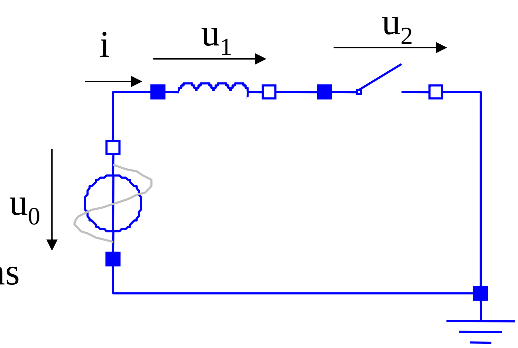



```
model VanDerPol
  import Matrices.eigenValues;
  import Vectors.norm;
  parameter Real epsilon=2;
  Real y[2] (start={1,-1});
  Real eigenV[2,2];
  Real delta_t;
equation
  der(y[1]) = y[2];
  der(y[2]) = epsilon*(1 - y[1]*y[1])*y[2] - y[1];
  eigenV=eigenValues(
           [                    0,                   1;
            -2*epsilon*y[1]*y[2]-1, epsilon*(1-y[1]*y[1])]);
  delta_t=max(-2*eigenV[1,1]/(eigenV[1,:]*eigenV[1,:]),
              -2*eigenV[2,1]/(eigenV[2,:]*eigenV[2,:]));
end VanDerPol;
```

$$\Longrightarrow \quad \Delta t_{max} = 0.0326524$$

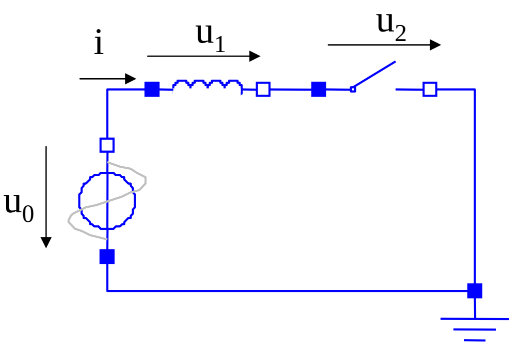
Event Handling Motivation and Numerical Issues

- Need of discontinuous components
  - Detailed modeling too complex
  - Model parameters are not known for detailed models
  - Non-ideal approximation leads to stiff systems
  - Speed up simulation time (omit stiffness)



- Numerical problems at events
  - ODE solver based on polynomial approximation
  - Usually some signals are not continuous or differentiable
  - Fixed step size leads to bad approximation of events
  - Varying higher-index problems can occur at events

Example: Non-Ideal Switch Stiffness Problem



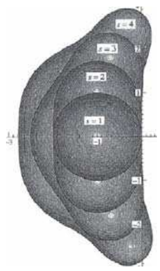
$$u_0 = A\sin(\omega t) \qquad \longrightarrow \qquad L\frac{di}{dt} = A\sin(\omega t) - R\,i$$
$$u_2 = Ri \qquad \downarrow$$
$$L\frac{di}{dt} = u_1 \qquad \boxed{\lambda = -\frac{R}{L}}$$
$$u_0 = u_1 + u_2$$

Non-ideal switch:

closed: $u_2 = R\,i$, $R=1e\text{-}5$

open: $G\,i = u_2$, $G=1e\text{-}5$

$$u_0 = A\sin(\omega t) \qquad \longrightarrow \qquad L\frac{di}{dt} = A\sin(\omega t) - \frac{i}{G}$$
$$i = Gu_2 \qquad \downarrow$$
$$L\frac{di}{dt} = u_1 \qquad \boxed{\lambda = -\frac{1}{L\cdot G}}$$
$$u_0 = u_1 + u_2$$

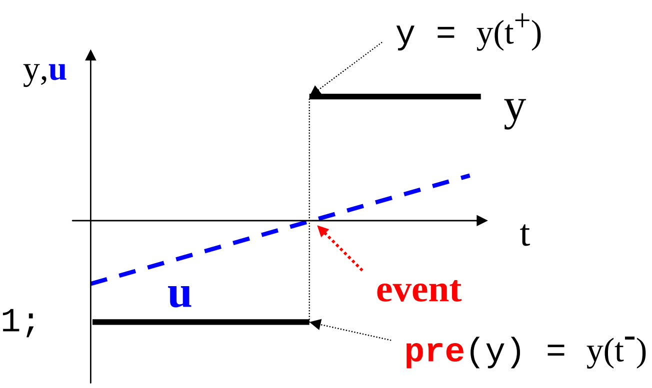


If the switch is open, system is stiff!

Continuous Variables and the if-Statement

- State events
  - zero-crossing function
  - i.e. bi-section method between adjacent time points

```
y = if u>0 then 1 else -1;
```



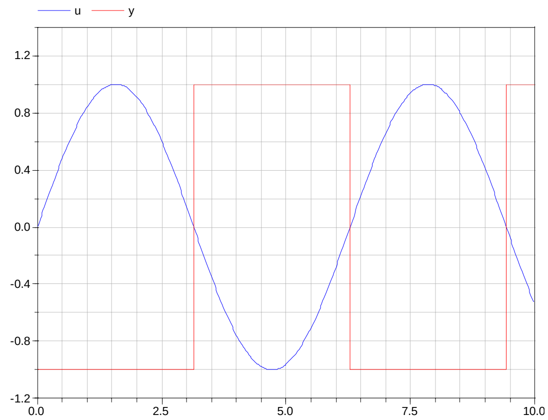
- Time events
  - no zero-crossing function necessary
  - special treatment during compile time possible

```
y = if time>.5 then 1 else -1;
```

Principle Strategy at Events (Simple Example)

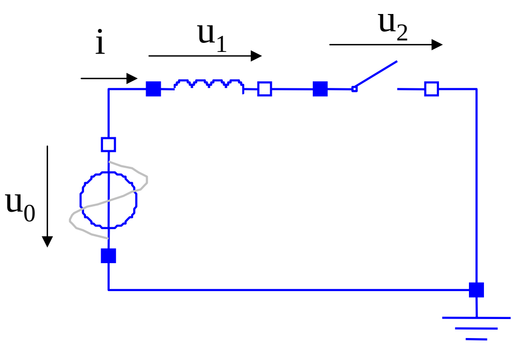
- Standard-solution method
  - Stop integration at event time
  - Do necessary adjustments
    - change input signals and/or discrete variables, re-initialize states
  - Restart integration routine
    - start-up of multi-step methods, initial step-size
- Continuous time integration between events
  - Appropriate step size control
- Store signals twice at event time
  - Variables before and after the event

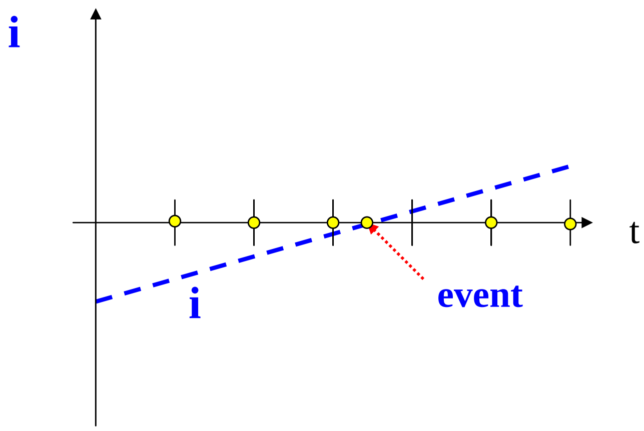
```
model StateEvent
  import Modelica.Math.sin;
  Real u;
  Real y;
equation
  u = sin(time);
  y = if u<0 then 1 else -1;
end StateEvent
```



Real-Time Solution to Events

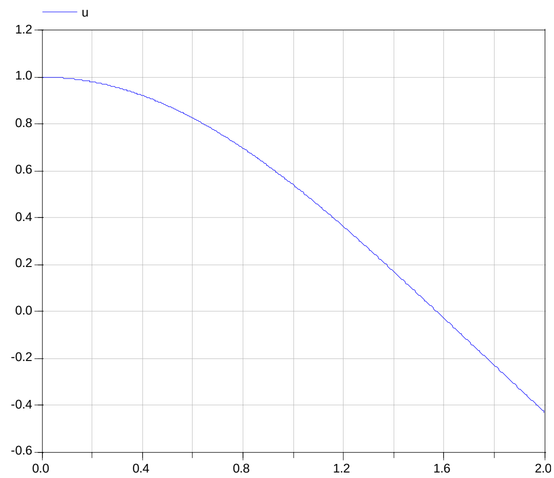
- One-step method with fixed step-size
  - Detect event (u crosses 0)
  - Approximate exact event time
    - use (linear) polynomial approximation of the signal
  - Restart integration routine
    - combine two equidistant steps





Influence Event-Handling

```
model SmoothEvent
  constant Real pi=Modelica.Constants.pi;
  Real x,z,y,u;
equation
  x = if time<pi/2 then cos(time) else pi/2-time;
  y = noEvent( if time<pi/2 then cos(time) else pi/2-time);
  z = smooth(1,if time<pi/2 then cos(time) else pi/2-time);
  u = smooth(1, noEvent(if time<pi/2 then cos(time) else pi/2-time));
end SmoothEvent
```



Semantical interpretation:

- Equation for x: triggers event
- Equation for y: no event handling
- Equation for z: may trigger event
  - may depend on the integration method
- Equation for u: no event handling

Sorting Equations Including if-Equations

- **Modelica model:**
  - Differential equations, algebraic equations

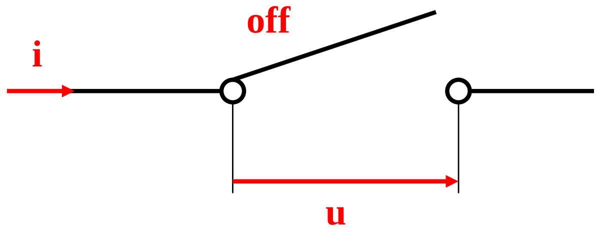
$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

  - if-equations
    - y = **if** <cond> **then** <expr1> **else** <expr2>;

- **Mathematical view:**
  - <cond> c is a boolean expression
    - may depend on a state or time event (Causality)
  - An if-equation is treated as one equation depending on <cond> c and the variables $w_1,\ldots w_n$ in <expr1> and $v_1, \ldots, v_m$ in <expr2>

$$0 = g\left(c, w_1, \ldots, w_n, v_1, \ldots, v_m, \underline{p}\right)$$

Example: Ideal Electrical Switch



The ideal switch is described by the following two equations:

off = true: **i = 0**

off = false: **u = 0**

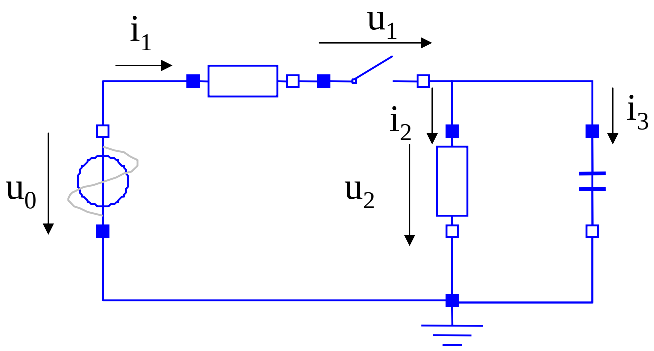
Corresponding Modelica code:

```
0 = if off then i else u;
```

Equivalent description (**off = 0/1**):

$0 = \mathit{off}\ i + (1 - \mathit{off})\ u \longrightarrow 0 = f(i, u, \mathit{off})$

Example: Ideal Electrical Switch



$$u_0 = A\sin(\omega t)$$
$$u_0 = R_1 \cdot i_1 + u_1 + u_2$$
$$0 = \mathit{off} \cdot i_1 + (1 - \mathit{off}) \cdot u_1$$
$$i_2 = R_2 / u_2$$
$$i_3 = i_1 - i_2$$
$$C\dot{u}_2 = i_3$$

Linear system of two equations:

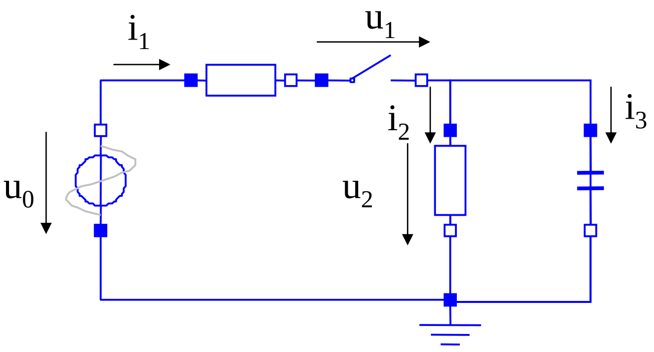
$$\begin{aligned} u_0 &= R_1 \cdot i_1 + u_1 + u_2 \\ 0 &= \mathit{off} \cdot i_1 + (1 - \mathit{off}) \cdot u_1 \end{aligned} \longrightarrow \begin{bmatrix} R_1 & 1 \\ \mathit{off} & (1 - \mathit{off}) \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} u_0 - u_2 \\ 0 \end{bmatrix}$$

Two possible solutions:

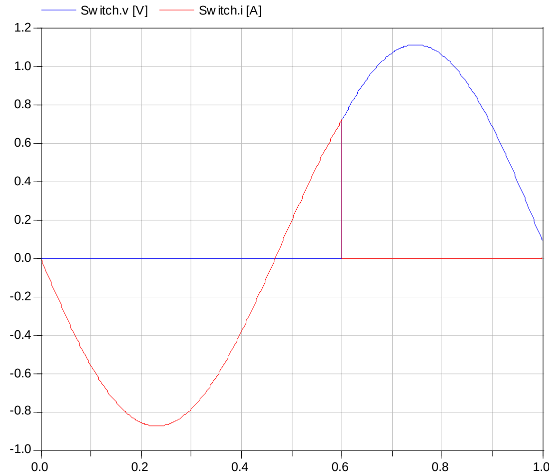
$$\mathit{off} = 1 \ : \quad i_1 = 0, \quad u_1 = u_0 - u_2$$

$$\mathit{off} = 0 \ : \quad u_1 = 0, \quad i_1 = (u_0 - u_2) / R_1$$

Example: Ideal Electrical Switch



$$u_0 = A\sin(\omega t)$$
$$u_0 = R_1 \cdot i_1 + u_1 + u_2$$
$$0 = off \cdot i_1 + (1 - off) \cdot u_1$$
$$i_2 = R_2 / u_2$$
$$i_3 = i_1 - i_2$$
$$C\dot{u}_2 = i_3$$

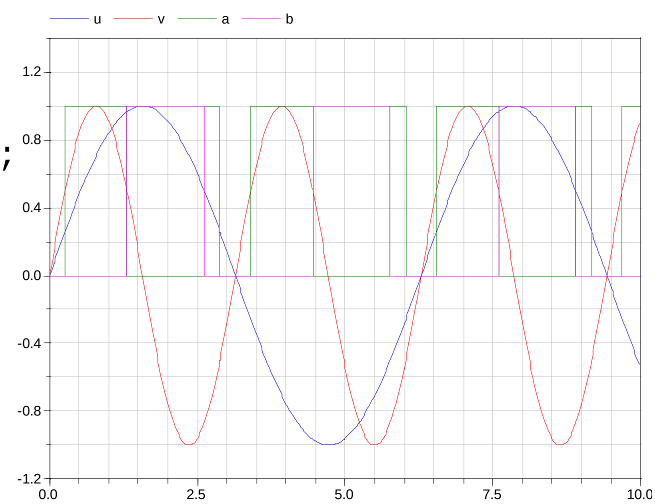


Unrealistic behavior :

- The current can not jump to zero
- Usually an arc keeps current flowing
- Current must be close to zero
- Additional logic necessary
  - needs **when**-equations
    (example will be continued)

Algebraic Loops Involving Integer and/or Boolean Variables

```
model AlgebraicLoopBoolean
  Boolean a(start=true);
  Boolean b;
  Real u;
  Real v;
equation
  u = sin(time);
  v = sin(2*time);
  b = not pre(a) and abs(u)>0.5;
  a = not b and abs(v)>0.5;
end AlgebraicLoopBoolean
```



Algebraic loops involving boolean variables must be handled by the user!

- Break loops using the **pre** operator

Discrete Variables and the when-Statement

Additional equations can be declared at an event using the **when**-statement. These equations are de-activated during the continuous integration.

```
when  <condition>  then
   <equations>
end when;
```

When `<condition>` becomes true, `<equations>` are calculated.

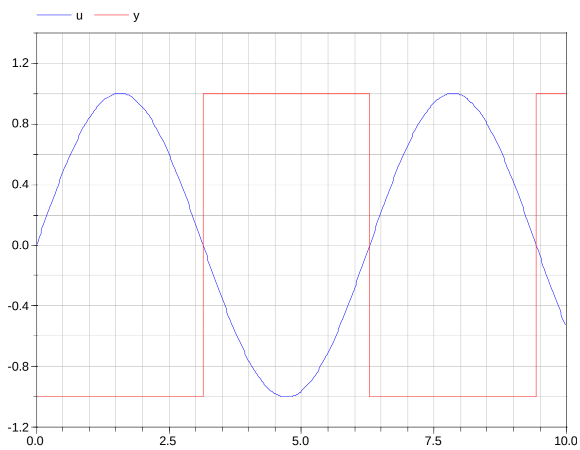
Equivalent formulation:

```
if edge(<condition>)  then
    <equations>
end if;
```

`<condition>` may not depend on a **`noEvent()`** Operator. Therefore, only at event points the equations within the when clause are evaluated

Re-writing State Events to Time Events

```
model HandleEvents
  constant Real PI=Modelica.Constants.pi;
  Real u;
  Real y;
  Boolean flag(start=false);
equation
  u = Modelica.Math.sin(time);
  when sample(PI,PI) then
    flag = not pre(flag);
  end when;
  y = if flag then 1 else -1;
//y = if u<0  then 1 else -1;
end HandleEvents
```



Real-time efficiency:

- If possible use time-events instead of state-events

Sorting Equations Including when-Statements

- **Modelica model:**
  - Differential equations, algebraic equations, and if-equations

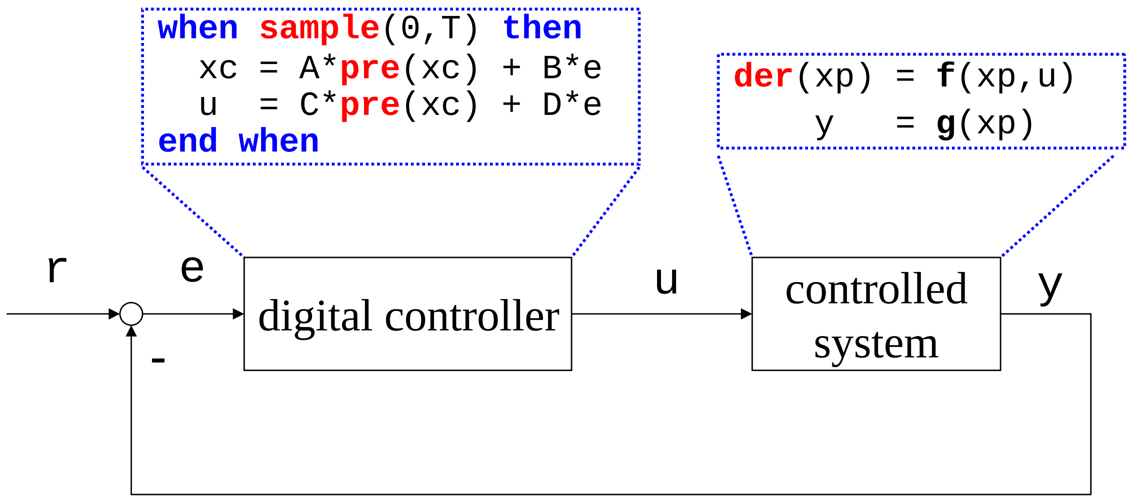
$$\underline{0} = \underline{f}\left(t, \underline{\dot{x}}(t), \underline{x}(t), \underline{y}(t), \underline{u}(t), \underline{p}\right)$$

  - Discrete equations only active at events
    - `when <cond> then <equations> end when;`

- **Mathematical view:**
  - Same number of equations and unknowns at each time point
  - Different number of equations and unknowns during event

- **Algorithmic strategy:**
  - Sorting is based on all equations

Example: Digital Controller




```
y := g(xp);
e := r - y;
when sample(0,T) then
  xc := A*pre(xc) + B*e;
  y  := C*pre(xc) + D*e;
end when
der(xp) := f(xp,u);
```

Order of evaluation is correct for the sample time points as well as during continued time integration!

(xp and **pre**(xc) are known)

Sorting Equations of Hybrid Models

- IMPORTANT:
  Each discrete variable is determined by one equation
  - Example:

```
when h1>0 then
   openValve = true;
end when;

when h2<2 then
   openValve = false;
end when;
```

  - Not allowed since two equations for one variable `openValve`
  - Combine conditions `h1>0` and `h2<2`

Sorting Equations of Hybrid Models

Re-writing event equations

```
when h1>0 or h2<2 then
   openValve = if edge(h1 > 0) then true else false;
end when;
```

Semantical interpretation:
If one condition becomes true, (i.e. `h1 > 0`), no further event will happen not even when the second condition (`h2 < 2`) becomes true.

Boolean expression (`h1>0 or h2<2`) will not change anymore!

This is not the desired logic!

Sorting Equations of Hybrid Models

Desired logic can be described by

```
when {h1>0, h2<2}  then
  openValve = if edge(h1 > 0) then true else false;
end when;
```

or even shorter:

```
when {h1>0, h2<2}  then
  openValve = edge(h1 > 0);
end when;
```

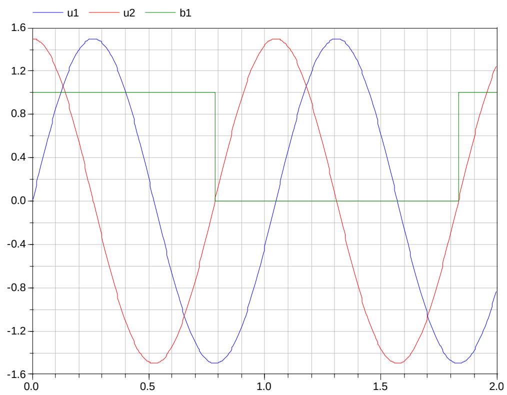
```
when {expr1, expr2, expr3, ...} then
```

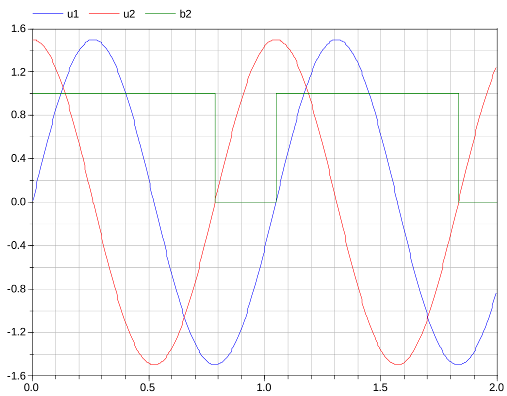
Semantical interpretation:

- Triggers an event, if one of the listed boolean expression becomes true

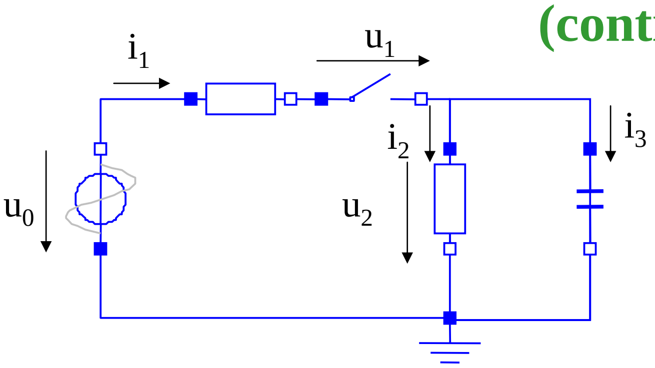
Example: Difference of „or“ and {...,...}

```
model whendemo3
   parameter Real A=1.5, w=6;
   Real     u1, u2;
   Boolean b1, b2;
equation
   u1 = A*sin(w*time) + 1.e-10;
   u2 = A*cos(w*time);
   when u1 > 0 or u2 > 0 then
      b1 = not pre(b1);
   end when;
   when {u1 > 0, u2 > 0} then
      b2 = not pre(b2);
   end when;
end whendemo3;
```





Example: Ideal Electrical Switch (continued)

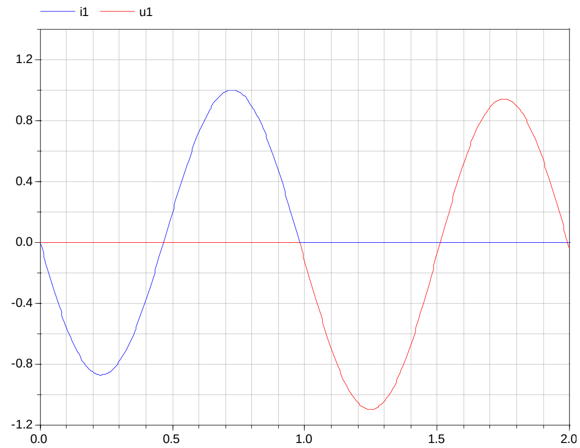


$$u_0 = A\sin(\omega t)$$
$$u_0 = R_1 \cdot i_1 + u_1 + u_2$$
$$0 = off \cdot i_1 + (1 - off)\cdot u_1$$
$$i_2 = R_2 / u_2$$
$$i_3 = i_1 - i_2$$
$$C\dot{u}_2 = i_3$$

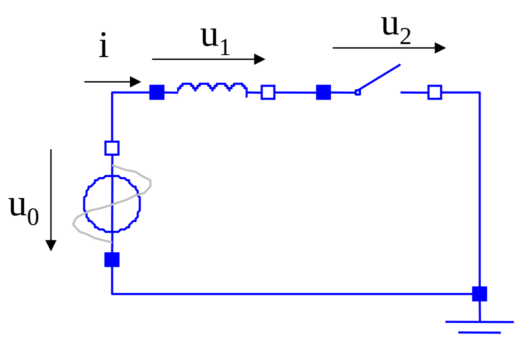
Introduce additional logic :

- Algebraic loop with boolean variable

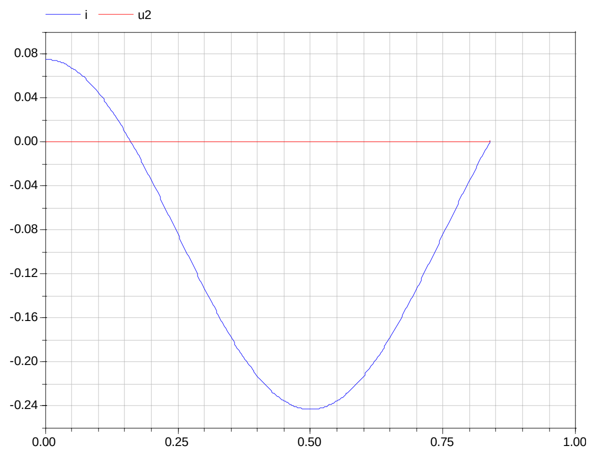
```
when (time>t0) then
  sign_i = if (i1>0) then 1 else -1;
end when;
when (time>t0 and sign_i*i1<0) then
  off = true;
end when;
0 = if pre(off) then i1 else u1;
```



Example: Ideal Electrical Switch (continued)



$$u_0 = A\sin(\omega t)$$
$$0 = off \cdot i + (1 - off)\cdot u_2$$
$$L\frac{di}{dt} = u_1$$
$$u_0 = u_1 + u_2$$



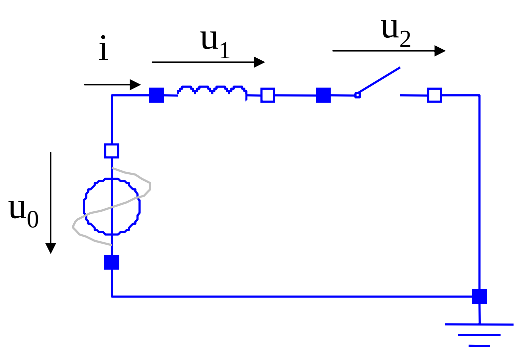
Causality (off = 0/1):

- i is a state and is therefore known

$$u_2 = \frac{-off \cdot i}{1 - off}$$

Only solvable, iff *off*=0

Example: Ideal Electrical Switch (continued)



$$u_0 = A\sin(\omega t)$$

$$0 = off \cdot \frac{di}{dt} + (1 - off) \cdot u_2$$

$$L\frac{di}{dt} = u_1$$

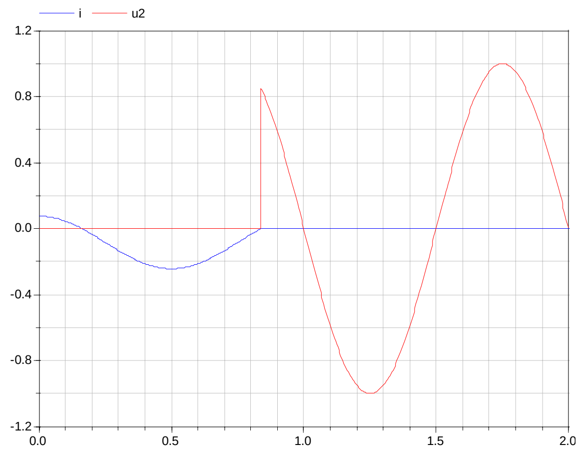
$$u_0 = u_1 + u_2$$

Varying higher-index problem

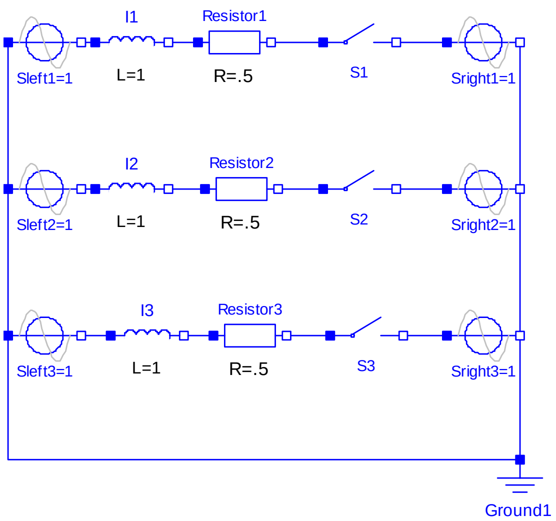
$off = 0 \quad : \quad u_2 = 0$ index=1

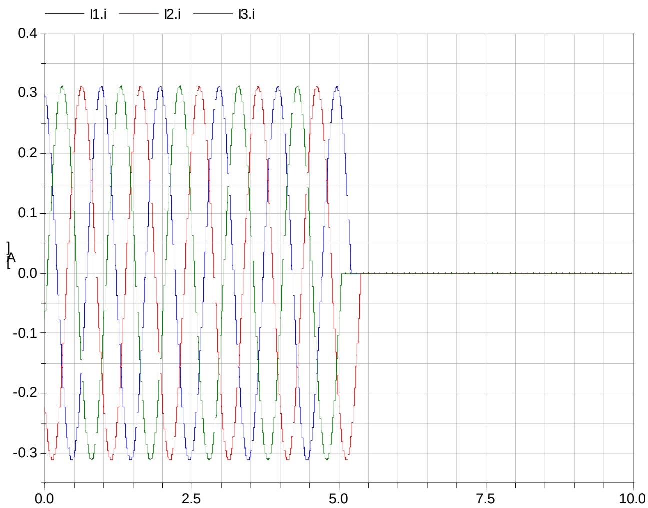
$off = 1 \quad : \quad i = 0$ index=2

$$\frac{di}{dt} = 0$$



Example: 3-Phase Electrical System (continued)



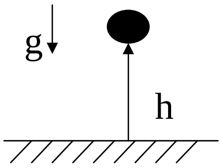


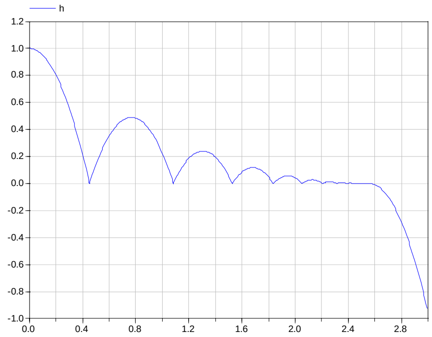
Switch works also for the 3-Phase electrical system

- Introducing dq0-reference frame still very efficient
- `i_dq0` used as state
- Initialization: `der(i_dq0)={0,0,0};`

Numerical Issues

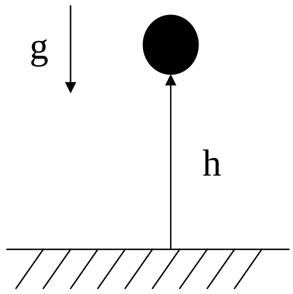
- Standard-solution method
  - Stop integration at event time
  - Do necessary adjustments
  - Restart integration routine
- Due to numerical errors the system can get into a non-physical state
  - Height h is negative in the bouncing ball example
- Solution to this problem
  - Changes in the settings (accuracy) for the solver routine may help
  - Different integration methods may lead to different results
  - Adapt logic to circumvent numerical problems (**Most Efficient**)






```
model bouncingBall
  parameter Real e=0.7;
  parameter Real g=9.81;
  Real h(start=1);
  Real v;
equation
  der(h) = v;
  der(v) = -g;
  when h < 0 then
    reinit(v, -e*pre(v));
  end when;
end bouncingBall;
```


Numerical Issues
Solution to the Bouncing Ball




```
model bouncingBall
  parameter Real e=0.7;
  parameter Real g=9.81;
  Real h(start=1);
  Real v;
  Boolean flying(start=true)
  Boolean impact;
  Real v_new;
equation
  der(h) = v;
  der(v) = if flying then -g else 0;
  impact = h < 0;
  when {impact,h < 0 and v < 0} then
    v_new = if edge(impact) then -e*pre(v)
                            else 0;
    flying = v_new > 0;
    reinit(v, v_new);
  end when;
end bouncingBall;
```